

CLASS XI

1. Sets

1. (d) Since, intelligency is not defined for students in a class i.e., Not a well defined collection.

2. (a) Let B, H, F denote the sets of members who are on the basketball team, hockey team and football team respectively.

Then we are given $n(B) = 21, n(H) = 26, n(F) = 29$

$n(H \cap B) = 14, n(H \cap F) = 15, n(F \cap B) = 12$

and $n(B \cap H \cap F) = 8$.

We have to find $n(B \cup H \cup F)$.

To find this, we use the formula

$$n(B \cup H \cup F) = n(B) + n(H) + n(F)$$

$$-n(B \cap H) - n(H \cap F) - n(F \cap B) + n(B \cap H \cap F)$$

Hence,

$$n(B \cup H \cup F) = (21 + 26 + 29) - (14 + 15 + 12) + 8 = 43$$

Thus these are 43 members in all.

3. (b) $n(A) = 40\% \text{ of } 10,000 = 4,000$

$$n(B) = 20\% \text{ of } 10,000 = 2,000$$

$$n(C) = 10\% \text{ of } 10,000 = 1,000$$

$$n(A \cap B) = 5\% \text{ of } 10,000 = 500$$

$$n(B \cap C) = 3\% \text{ of } 10,000 = 300$$

$$n(C \cap A) = 4\% \text{ of } 10,000 = 400$$

$$n(A \cap B \cap C) = 2\% \text{ of } 10,000 = 200$$

We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$

$$= n(A) - n[A \cap (B \cup C)] = n(A) - n[(A \cap B) \cup (A \cap C)]$$

$$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$$

$$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300.$$

4. (d) Let set P be the families who own a phone and set C be the families who own a car.

$$n(P) = 25\%, n(C) = 15\%,$$

$$n(P^c \cap C^c) = 65\% \text{ and } n(P \cap C^c) = 35\%$$

Now, $n(P \cap C) = n(P) + n(C) - n(P \cup C) = 25 + 15 - 35 = 15\%$

$$\Rightarrow x \times 5\% = 2000 \Rightarrow x = 40,000$$

5. (d) $n(M) = 23, n(P) = 24, n(C) = 19$

$$n(M \cap P) = 12, n(M \cap C) = 9, n(P \cap C) = 7$$

$$n(M \cap P \cap C) = 4$$

We have to find $n(M \cap P' \cap C'), n(P \cap M' \cap C')$,

$$n(C \cap M' \cap P')$$

$$\text{Now } n(M \cap P' \cap C') = n[M \cap (P \cup C)']$$

$$= n(M) - n(M \cap (P \cup C))$$

$$= n(M) - n[(M \cap P) \cup (M \cap C)]$$

$$= n(M) - n(M \cap P) - n(M \cap C) + n(M \cap P \cap C)$$

$$= 23 - 12 - 9 + 4 = 27 - 21 = 6$$

$$n(P \cap M' \cap C') = n[P \cap (M \cup C)']$$

$$= n(P) - n[P \cap (M \cup C)] = n(P) - n[(P \cap M) \cup (P \cap C)]$$

$$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$$

$$= 24 - 12 - 7 + 4 = 9$$

$$n(C \cap M' \cap P') = n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M)$$

$$= 19 - 7 - 9 + 4 = 23 - 16 = 7.$$

6. (c) Number of proper subsets of the set $\{1, 2, 3\} = 2^3 - 2 = 6$.

7. (b) Since $x^2 + 1 = 0$, gives $x^2 = -1 \Rightarrow x = \pm i$

$\therefore x$ is not real but x is real (given)

\therefore No value of x is possible.

8. (a) $x^2 = 16 \Rightarrow x = \pm 4$

$$2x = 16 \Rightarrow x = 3$$

There is no value of x which satisfies both the above equations. Thus, $A = \emptyset$.

9. (c) Number of subsets of $A = {}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$.

10. (c) Number of proper subsets of the set $\{1, 2, 3\} = 2^3 - 2 = 6$.

11. (b) $B \cap C = \{4\}$, $\therefore A \cup (B \cap C) = \{1, 2, 3, 4\}$.

12. (c) Since, $y = e^x$ and $y = x$ do not meet for any $x \in R$

$\therefore A \cap B = \emptyset$

13. (c) $n(A^c \cap B^c) = n[(A \cup B)^c] = n(U) - n(A \cup B)$

$$= n(U) - [n(A) + n(B) - n(A \cap B)]$$

$$= 700 - [200 + 300 - 100] = 300.$$

14. (b) $N_5 \cap N_7 = N_{35}$,

[$\because 5$ and 7 are relatively prime numbers].

15. (c) $n(A \times B) = pq$

16. (d) It is fundamental concept.

17. (b) Since $\frac{1}{y} \neq 0, \frac{1}{y} \neq 2, \frac{1}{y} \neq -\frac{2}{3}$, [$\because y \in N$]

$\therefore \frac{1}{y}$ can be 1, [$\because y$ can be 1].

18. (d) It is obvious.

19. (a) Minimum value of $n = 100 - (30 + 20 + 25 + 15) = 100 - 90 = 10$.

20. (b) $A \cup B = \{1, 2, 3, 8\}; A \cap B = \{3\}$.

$$(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}.$$

2. Relations and Functions

1. (d) We have, $f(g)(x)) = x$

$$\begin{aligned} f(3^{10}x - 1) &= 2^{10}(3^{10}x - 1) + 1 = x \\ \Rightarrow 2^{10}3^{10}x - 2^{10} + 1 &= x \Rightarrow x(2^{10}3^{10} - 1) = 2^{10} - 1 \\ \Rightarrow x &= \frac{2^{10} - 1}{2^{10}3^{10} - 1} \Rightarrow x = \frac{2^{10}(1 - 2^{-10})}{2^{10}(3^{10} - 2^{-10})} \Rightarrow x = \frac{1 - 2^{-10}}{3^{10} - 2^{-10}} \end{aligned}$$

2. (b) $2\{x\}^2 - 3\{x\} + 1 \geq 0$

$$\Rightarrow \{x\} \leq \frac{1}{2} \Rightarrow x \in \left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$$

3. (b) Since R is an equivalence relation on set A , therefore $(a, a) \in R$ for all $a \in A$. Hence, R has at least n ordered pairs. i.e., greater than or equal to n

4. (d) Since $f(x)$ is an odd function,

$$\left[\frac{x^2}{a}\right] = 0 \text{ for all } x \in [-10, 10] \Rightarrow 0 \leq \frac{x^2}{a} < 1 \text{ for all } x \in [-10, 10] \Rightarrow a > 100$$

$$\begin{aligned} 5. (c) f : N \rightarrow N, \quad f(x) &= x - (-1)^n \\ &= x + (-1)^{n+1} \end{aligned}$$

$$\Rightarrow f(2m) = 2m - 1$$

$f(2m + 1) = 2m + 2 \Rightarrow f$ is one-one & onto.

6. (a) To be an equivalence relation, the relation must be reflexive, symmetric and transitive.

$T = \{(x,y) : x - y \in Z\}$ is

reflexive - for $(x,x) \in Z$ i.e. $x - x = 0 \in Z$

symmetric - for $(x,y) \in Z \Rightarrow x - y \in Z$

$\Rightarrow y - x \in Z$ i.e. $(y,x) \in Z$

transitive - for $(x,y) \in Z$ and $(y,w) \in Z$

$\Rightarrow x - y \in Z$ and $y - w \in Z$, giving $x - w \in Z$ i.e. $(x,w) \in Z$.

$\therefore T$ is an equivalence relation on R .

reflexive for $(x,x) \in S$ would imply $x = x + 1$

Since, $0 < x < 1$

So, for $x = 1 \Rightarrow 1 \neq 1 + 1$

$$1 \neq 2$$

Thus S is not an equivalence relation.

7. (b) For any integer n , we have $n | n \Rightarrow nRn$

So, nRn for all $n \in Z \Rightarrow R$ is reflexive

Now $6|2$ but $6 + 2, \Rightarrow (6, 2) \in R$ but $(2, 6) \notin R$

So, R is not symmetric.

Let $(m, n) \in R$ and $(n, p) \in R$.

$$\text{Then } \begin{cases} (m, n) \in R \Rightarrow m | n \\ (n, p) \in R \Rightarrow n | p \end{cases} \Rightarrow m | p \Rightarrow (m, p) \in R$$

So, R is transitive.

Hence, R is reflexive and transitive but it is not symmetric.

8. (d) If n is odd, let $n = 2k - 1$

$$\text{Let } f(2k_1 - 1) = f(2k_2 - 1) \Rightarrow \frac{2k_1 - 1 - 1}{2} = \frac{2k_2 - 1 - 1}{2} \Rightarrow k_1 = k_2$$

$f(n)$ is one-one function if n is odd

Again, if $n = 2k$ (i.e. n is even)

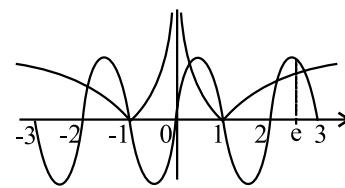
Let $f(2k_1) = f(2k_2)$

$$\Rightarrow -\frac{2k_1}{2} = -\frac{2k_2}{2} \Rightarrow k_1 = k_2 \Rightarrow f(n)$$
 is one-one if n is even

range (f) = Z = codomain $\Rightarrow f$ is onto.

Now all such functions which are either increasing or decreasing in the states domain are said to be onto function. Finally $f(n)$ is one-one function.

9. (d) $\sin \pi x = |\ell n |x||$



Number of solution is 6

10. (c) It is obvious.

11. (c) Let $x = 0 = y \Rightarrow f(0) = 0$

and $x = 1, y = 0 \Rightarrow f(1 + 0) = f(1) + f(0) = 7$ (given)

$x = 1, y = 1 \Rightarrow f(1 + 1) = 2f(1) = 2(7)$

$$\Rightarrow f(2) = 2(7)$$

$$x = 1, y = 2 \quad \therefore f(3) = f(1) + f(2) = 7 + 2(7) = 3(7)$$

and so on.

$$\begin{aligned} \therefore \sum_{r=1}^n r(r) &= f(1) + f(2) + f(3) + \dots + f(n) \\ &= 7(1 + 2 + 3 + \dots + n) = \frac{7n(n+1)}{2} \end{aligned}$$

12. (a) Since $1 + a.a = 1 + a^2 > 0, \forall a \in S$, $\therefore (a, a) \in R$

$\therefore R$ is reflexive.

Also $(a, b) \in R \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$,

$\therefore R$ is symmetric.

$(1, \frac{1}{2}) \in R$ and $(\frac{1}{2}, -1) \in R$ but $(1, -1) \notin R$ as $1 - 1 > 0$

$\therefore (a, b) \in R$ and $(b, c) \in R$ need not imply $(a, c) \in R$. Hence, R is not transitive.

13. (c) Here $(3, 3), (6, 6), (9, 9), (12, 12)$, [Reflexive];

$(3, 6), (6, 12), (3, 12)$, [Transitive].

Hence, reflexive and transitive only.

14. (c) $f(x) = \frac{\cos(\sin nx)}{\tan\left(\frac{x}{n}\right)}$

period of $\cos(\sin nx)$ is $\frac{\pi}{n}$ and of $\tan\left(\frac{x}{n}\right)$ will be $n\pi$, so period of $f(x)$ will be $n\pi \Rightarrow x = 6$

15. (a) $n(A \times A) = n(A) \cdot n(A) = 3^2 = 9$

So, the total number of subsets of $A \times A$ is 2^9 and a subset of $A \times A$ is a relation over the set A .

16. (d) R_4 is not a relation from X to Y , because $(7, 9) \in R_4$ but $(7, 9) \notin X \times Y$.

17. (a) Since $(1, 1); (2, 2); (3, 3) \in R$ therefore R is reflexive. $(1, 2) \in R$ but $(2, 1) \notin R$, therefore R is not symmetric. It can be easily seen that R is transitive.

18. (b) Since

$\therefore xRy, yRz \Rightarrow xRz$, \therefore Relation is transitive,

$\therefore x < y$ does not give $y < x$,

\therefore Relation is not symmetric.

Since x, x does not hold, hence relation is not reflexive.

19. (d) Given, $xRy \Rightarrow x$ is relatively prime to y .

\therefore Domain of $R = \{2, 3, 4, 5\}$.

20. (c) $\because R = \{(x, y) | x, y \in Z, x^2 + y^2 \leq 4\}$

$\therefore R = \{(-2, 0), (-1, 0), (-1, 1), (0, -1), (0, 1), (0, 2), (0, -2)$

$(1, 0), (1, 1), (2, 0)\}$

Hence, Domain of $R = \{-2, -1, 0, 1, 2\}$.

3. Trigonometric Functions

1. (b) We know that; $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{y}{1 - \tan A \tan B}, \text{ where } y = \tan A + \tan B$$

$$\Rightarrow \tan A \tan B = 1 - \sqrt{3}y$$

$$\text{Also A.M.} \geq \text{G.M.} \Rightarrow \frac{\tan A + \tan B}{2} \geq \sqrt{\tan A \tan B}$$

$$\Rightarrow y \geq 2\sqrt{1 - \sqrt{3}y} \Rightarrow y^2 \geq 4 - 4\sqrt{3}y \Rightarrow y^2 + 4\sqrt{3}y - 4 \geq 0$$

$$y \leq -2\sqrt{3} - 4 \text{ or } y \geq -2\sqrt{3} + 4$$

$(y \leq 2\sqrt{3}) - 4$ is not possible as $\tan A, \tan B > 0$

2. (a) $\cot A = \tan(90^\circ - A)$

$$\cot A \tan A = 1$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan 10^\circ = \tan(50^\circ - 40^\circ) = \frac{\tan 50^\circ - \tan 40^\circ}{1 + \tan 50^\circ \tan 40^\circ}$$

$$\Rightarrow \tan 50^\circ - \tan 40^\circ = (1 + \tan 50^\circ \tan 40^\circ) \tan 10^\circ$$

$$= (1 + \tan 50^\circ \cot 50^\circ) \tan 10^\circ$$

$$= (1+1) \tan 10^\circ$$

$$= 2 \tan 10^\circ$$

$$\Rightarrow k = 2$$

3. (b) Since $\frac{a}{b} = \frac{2+\sqrt{3}}{1} \Rightarrow \angle A > \angle B$

So only option (b) & (d) can be correct.

$$\frac{a}{b} = \frac{\sin 105^\circ}{\sin 15^\circ} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} \text{ which is true}$$

$$4. (a) \frac{1 + \sin 2x}{1 - \sin 2x} = \frac{\sin\left(\frac{\pi}{2}\right) + \sin 2x}{\sin\left(\frac{\pi}{2}\right) - \sin 2x}$$

$$= \cot\left(\frac{\pi}{4} + x\right) \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + x\right)\right)$$

$$= \cot^2\left(\frac{\pi}{4} + x\right)$$

5. (c) LHS = $\frac{1 + \sin \theta - \cos \theta}{1 + \cos \theta + \sin \theta}$

$$= \frac{\sin \theta (1 + \sin \theta - \cos \theta)}{\sin \theta (1 + \cos \theta + \sin \theta)}$$

$$= \frac{\sin \theta + \sin^2 \theta - \sin \theta \cos \theta}{\sin \theta (1 + \cos \theta + \sin \theta)}$$

$$= \frac{\sin \theta - \sin \theta \cos \theta + (1 - \cos^2 \theta)}{\sin \theta (1 + \cos \theta + \sin \theta)}$$

$$= \frac{\sin \theta (1 - \cos \theta) + (1 - \cos^2 \theta)}{\sin \theta (1 + \cos \theta + \sin \theta)}$$

$$= \frac{(1 - \cos \theta)(\sin \theta + 1 + \cos \theta)}{\sin \theta (1 + \cos \theta + \sin \theta)}$$

$$= \frac{1 - \cos \theta}{\sin \theta}$$

$$= \frac{2 \sin^2\left(\frac{\theta}{2}\right)}{2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

6. (b) y has four solutions i.e. $-\tan A, \cot A,$

$$\tan\left(\frac{\pi}{4} + A\right), -\cot\left(\frac{\pi}{4} + A\right)$$

$$\text{As } A \in \left(\frac{7\pi}{8}, \frac{9\pi}{8}\right) \text{ so, } y = \cot A$$

7. (a) $2\sin^2 x + 5 \sin x - 3 = 0 \Rightarrow \sin x = \frac{1}{2}, \sin x \neq -3$

therefore $\sin x = \frac{1}{2}$, we know that each trigonometric function assumes same value twice in $0 \leq x \leq 360^\circ$.

In our problem $0^\circ \leq x \leq 540^\circ$. So number of values are 4 like $30^\circ, 150^\circ, 390^\circ, 510^\circ$.

8. (a) $\tan 20^\circ \tan 80^\circ \cot 50^\circ$

$$= \tan 20^\circ \tan 80^\circ \cot(90^\circ - 50^\circ)$$

$$= \tan 20^\circ \tan 80^\circ \tan 40^\circ$$

$$= \tan 20^\circ \tan 40^\circ \tan 80^\circ$$

Now,

$$\tan 40^\circ \cdot \tan 80^\circ \cdot \tan 20^\circ$$

$$= \frac{(\sin 40^\circ \sin 80^\circ) \sin 20^\circ}{(\cos 40^\circ \cos 80^\circ) \cos 20^\circ}$$

$$= \frac{(2 \sin 40^\circ \sin 80^\circ) \sin 20^\circ}{(2 \cos 40^\circ \cos 80^\circ) \cos 20^\circ}$$

$$= \frac{(\cos 40^\circ - \cos 120^\circ) \sin 20^\circ}{(\cos 120^\circ + \cos 40^\circ) \cos 20^\circ}$$

$$= \frac{(2 \cos 40^\circ + 1) \sin 20^\circ}{(2 \cos 40^\circ - 1) \cos 20^\circ}$$

$$= \frac{2 \cos 40^\circ \sin 20^\circ + \sin 20^\circ}{2 \cos 40^\circ \cos 20^\circ - \cos 20^\circ}$$

$$= \frac{\sin 60^\circ - \sin 20^\circ + \sin 20^\circ}{\cos 60^\circ + \cos 20^\circ - \cos 20^\circ}$$

$$= \frac{\sin 60^\circ}{\cos 60^\circ} \\ = \tan 60 = \sqrt{3}.$$

9. (a) $\cos x \cdot \cos(6x) = -1$

$$\Rightarrow \cos(6x) = -\frac{1}{\cos x}$$

$$\Rightarrow \cos(6x) = -\frac{1}{\cos x} = -\sec x \quad \dots\dots(i)$$

We know that;

$$\cos \theta \in [-1, 1] \text{ and } \sec \theta \in (-\infty, -1] \cup [1, \infty]$$

Hence $\cos \theta$ and $\sec \theta$ can only be equal if both are equal to 1 or both are equal to -1 .

Hence (i) can hold if and only if

$$\cos(6x) = 1 \text{ and } \cos x = -1 \text{ or } \cos(6x) = -1 \text{ and } \cos x = 1$$

Now $\cos x = -1$ at odd multiples of π and $\cos x = 1$ at even multiples of π .

If x is even multiple of π , then $6x$ will also be even multiple of π .

Hence $\cos(6x) = -1$ and $\cos x = -1$ is not possible because if $\cos x = 1$, then $\cos(6x) = 1$ too

So it leaves us with

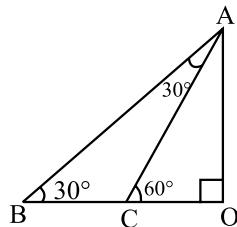
$$\cos(6x) = 1 \text{ and } \cos x = -1$$

As stated above, $\cos x = -1$ at odd multiples of π

$$\Rightarrow x = (2n+1)\pi, \text{ when 'n' is any integer.}$$

10. (a)

11. (c) Breadth of river OC = AC $\cos 60^\circ = 40 \cos 60^\circ = 40 \times \frac{1}{2} = 20$ m



12. (b) $\sin x \cos y = 1$

$$\Rightarrow \sin x = 1, \cos y = 1 \text{ or } \sin x = -1, \cos y = -1$$

$$\cos x = 1 \text{ at } 0, \text{ and } 2\pi \text{ and } \sin x = 1 \text{ at } \frac{\pi}{2}$$

So, only 3 ordered pairs are possible.

13. (c) Adding the given equations we get

$$\cos x \cos y + \sin x \sin y = 1$$

$$\Rightarrow \cos(x-y) = 1$$

$$\Rightarrow x-y = 0, 2\pi$$

$$\Rightarrow x = y, y + 2\pi$$

So, 4 solutions are possible.

14. (a) $\tan(A-B) = 1$

$$\text{So } (A-B) = \frac{\pi}{4} \quad \dots\dots(1)$$

$$\text{And } \sec(A+B) = \frac{2}{\sqrt{3}}, \text{ then } \cos(A+B) = \frac{\sqrt{3}}{2}$$

$$\text{Or } (A+B) = \frac{\pi}{6} \quad \dots\dots(2)$$

Solving (1) and (2), we have

$$2A = \frac{10\pi}{24} \text{ or } A = \frac{5\pi}{24}$$

$$B = \frac{-\pi}{24}$$

As we want positive value, so as trigonometric equations are periodic

$$\text{So } 2\pi + \left(\frac{-\pi}{24}\right) = \frac{47\pi}{24} = B \text{ will satisfies the above equations.}$$

15. (c) $\sin \alpha + \sin \beta = -\frac{21}{65}$ and $\cos \alpha + \cos \beta = \frac{27}{65}$

squaring and adding we get

$$2(1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta) = \frac{(21)^2 + (27)^2}{(65)^2}$$

$$2[1 + \cos(\alpha - \beta)] = \frac{1170}{(65)^2}$$

$$\cos^2 \frac{(\alpha-\beta)}{2} = \frac{1170}{4 \times 65 \times 65} = \frac{130 \times 9}{(130) \times (130)} = \frac{9}{130}$$

$$\therefore \cos \frac{\alpha-\beta}{2} = \frac{3}{\sqrt{130}}$$

As $\pi < \alpha - \beta < 3\pi$ then $\cos \left(\frac{\alpha-\beta}{2}\right)$ = negative.

16. (a) $\tan 2\theta = \cot \theta \Rightarrow \tan 2\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$$2\theta = n\pi + \frac{\pi}{2} - \theta \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{6}$$

17. (d) $\sin x = \cos^2 x$

$$\therefore \cos^8 x + 2 \cos^6 x + \cos^4 x = \sin^4 x + 2 \sin^3 x + \sin^2 x \\ = (\sin x + \sin^2 x)^2 = 1$$

18. (a) $\sin \theta = \frac{1}{\sqrt{2}}$ and $\frac{\pi}{2} < \theta < \pi$

$$\text{then } \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\text{So } \frac{\sin \theta + \cos \theta}{\tan \theta} = \frac{\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4}}{\tan \frac{3\pi}{4}}$$

$$= \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{-1} = 0$$

$$\begin{aligned}
 19. (c) & \cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ \\
 &= \cos 20^\circ \cos 100^\circ - \cos 100^\circ \cos 40^\circ - \cos 40^\circ \cos 20^\circ \\
 &= \frac{1}{2} [\cos 120^\circ + \cos 80^\circ - \cos 140^\circ - \cos 60^\circ - \cos 60^\circ - \cos 20^\circ] \\
 &= \frac{1}{2} \left[-\frac{3}{2} + 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \right] = -\frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 20. (b) & \frac{1 - \cos 4\alpha}{\sec^2 2\alpha - 1} + \frac{1 + \cos 4\alpha}{\csc^2 2\alpha - 1} \\
 &= \frac{2 \sin^2 2\alpha \cdot \cos^2 2\alpha}{1 - \cos^2 2\alpha} + \frac{2 \cos^2 2\alpha \cdot \sin^2 2\alpha}{1 - \sin^2 2\alpha} \\
 &= 2 \cos^2 2\alpha + 2 \sin^2 2\alpha = 2
 \end{aligned}$$

4. Principle of Mathematical Induction

1. (a) $\forall n \in N, P(n): 3^{3n} < 2n + 1$

$$P(1): 26 = 2 \times 13$$

$$P(2): 726 = 2 \times 343$$

$$P(3): 19683 - 6 + 1 = 19678 = 2 \times 9839$$

2. (d) $\forall n \in N, P(n): 2.4^{2n+1} + 3^{3n+1}$

$$P(1) = 209 = 11.19$$

$$P(2) = 11.385$$

3. (b) $P(n): 3^n < n!$, $n \in N$

$P(1): 3^1 < 1$ is not true. $P(3): 3^3 < 3!$ is not true.

$P(6): 3^6 < 6!$ is not true.

$P(7): 3^7 < 7!$ is true.

4. (a) For each $n \in N$, $P(n): 3^{2n} - 1$

$$P(1) = 8, P(2) = 80 = 10.8$$

5. (d) $P(1)$ validity cannot be checked because statement $P(n)$ is given

6. (c) For each $n \in N$, $P(n): 2^{3n} - 7n - 1$

$$P(1) = 0, P(2) = 49, P(3) = 512 - 21 - 1 = 490 = 49.10$$

$$7. (c) A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\therefore A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

$$\text{Now } nA - (n \cdot 1)I = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} \cdot \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$$

8. (c) $\because \sqrt{n(n+1)} < \sqrt{(n+1)(n+1)}$

$$\text{i.e. } \sqrt{n(n+1)} < n+1 \quad \forall n \in N$$

Hence reason is true.

For $n = 2$ given result is true.

let it is true for $n = K \in N, K \geq 2$ then

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{K}} > \sqrt{K}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}}$$

$$= \frac{\sqrt{K(K+1)} + 1}{\sqrt{K+1}} > \frac{\sqrt{KK} + 1}{\sqrt{K+1}} = \sqrt{K+1}$$

(\therefore by reason $\sqrt{n(n+1)} < n+1 \Rightarrow \sqrt{n} < \sqrt{n+1}$)

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{K+1}} > \sqrt{K+1}$$

Hence assertion is true for every natural number $n \geq 2$.

$$9. (a) A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$P(1): A = A - (1-1)I = A \therefore P(1)$ is true.

$P(k)$ is true $\Rightarrow P(k+1)$ is true, $k \in N$

10. (b) For smallest positive integer n , $P(n): n! < \{(n+1)/2\}^n$

$P(1): 1 < 1$ isn't true, $P(2): 2 < 9/4$ is true. $P(3): 6 < 8$ is true
 $P(4)$ is true.

11. (c) $\forall n \in N, P(n) = 11^{n+2} + 12^{2n+1}$

$$P(1): 11^{1+2} + 12^{2+1} = 133 \times 23,$$

$$P(2): 11^{2+2} + 12^{4+1} = 1464 + 248832 = 263473 - 133 \times 1981$$

12. (d) $P(n): n^2 + n + 1 = n(n+1) + 1$

$P(1): 3$ which is true.

$P(n): n^2 + n + 1 = n(n+1) + 1$ which is always odd number

13. (b) For $n \in N, P(n) - 2^n(n-1)! < n^n$

$P(1)": 2 < 1$ is not true.

$P(2): 4 < 4$ is not true.

$P(3): 16 < 27$ is true.

Same as $P(4)$ is true.

14. (a) For each $n \in N, P(n): 10^{2n-1} + 1$

$$P(1) = 11,$$

$$P(2) = 1001 = 11.91$$

15. (b) $S(K) = 1 + 3 + 5 + \dots + (2K-1) = 3 + K^2$

$S(1)$ is not true

let $S(K)$ is true then

$$S(K+1) = 1 + 3 + 5 + \dots + (2K-1) + (2K+1)$$

$$= S(K) + (2K+1)$$

$$= 3 + K^2 + 2K + 1 = 3 + (K+1)^2$$

Hence $S(K) \Rightarrow S(K+1)$

16. (b) Since $1^2 + 2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \dots$

$$n(n+1)^2 \text{ terms} = \frac{n(n+1)^2}{2}, \text{ when } n \text{ is even}$$

When n is odd the n^{th} term of series will be n^2 in this case, $(n-1)$ is even

so for finding sum of first $(n \cdot 1)$ terms of the series, we replacing n by $(n \cdot 1)$ in the given formula.

$$\text{So sum of first } (n \cdot 1) \text{ terms} = \frac{(n-1)n^2}{2}$$

Hence sum of n terms of the series

$$= (\text{the sum of } (n-1) \text{ terms} + \text{the } n^{\text{th}} \text{ term})$$

$$= \frac{(n-1)n^2}{2} + n^2 = \frac{(n+1)n^2}{2}$$

17. (b)

18. (b)

19. (c)

20. (a)

5. Complex Numbers

1. (a) $\bar{z}_1 = \frac{z_1 \bar{z}_1}{z_1} = |z_1|^2 z_1^{-1}$

$$\Rightarrow \arg(z_1^{-1}) = \arg(\bar{z}_1) \Rightarrow \arg(z_2) \Rightarrow z_2 = kz_1^{-1} (k > 0)$$

2. (b) We have $x - 1 = i \Rightarrow (x - 1)^4 = 1$

$$(x - 1)^2 = -1$$

$$(x - 1)^3 = -1$$

On substituting, we get $x^4 - 4x^3 + 7x^2 - 6x + 3 = 1$

3. (d) $|iz + (3 - 4i)| \leq |iz| + |3 - 4i| = |z| + 5 < 4 + 5 = 9$

Hence (d) is the correct option.

4. (c) Let $z = x + iy$, then $z^2 + \bar{z}^2 = 2 \Rightarrow x^2 - y^2 = 1$, which represents a hyperbola.

5. (d) Putting $z = a + 2i$ in the given equation and comparing imaginary parts, we get $a^2 + 4 = a^2$, which is not possible.

6. (a) $A + iB = \frac{1 - i\alpha}{1 + i\alpha} \Rightarrow A - iB = \frac{1 + i\alpha}{1 - i\alpha}$

$$\Rightarrow (A + iB)(A - iB) = \frac{(1 - i\alpha)(1 + i\alpha)}{(1 + i\alpha)(1 - i\alpha)} = 1$$

$$A^2 + B^2 = 1$$

7. (b) $\sum_{r=0}^{n-1} \cos r\alpha = \operatorname{Re} \sum_{r=0}^{n-1} e^{\frac{i\pi}{n}} = \text{sum of the } n \text{ roots of unity} = 0$

8. (a) Since the diagonals are perpendicular to each other

$$\arg \frac{z_1 - z_2}{z_2 - z_4} = \pm \frac{\pi}{2} \Rightarrow (z_1 - z_3) = i k(z_2 - z_4)$$

9. (d) Putting $x = 1, \omega, \omega^2$ in $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, we get

$$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + a_3 + \frac{1}{4} + a_{2n},$$

$$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \frac{1}{4} + a_{2n}\omega^{2n},$$

$$\text{and } (1 + \omega^2 + \omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \frac{1}{4} + a_{2n}\omega^{4n}$$

Adding the above three equations and using $1 + \omega + \omega^2 = 0, \omega^3 = 1$ we get

$$3^n = 3(a_0 + a_3 + a_6 + \dots)$$

$$\Rightarrow a_0 + a_3 + a_6 + \dots = 3^{n-1}.$$

10. (d) We have

$$\text{Now } = \omega^3\alpha^2 + \omega^4\alpha\beta + \omega^2\alpha\beta + \omega^3\beta$$

$$= \alpha^2 + \beta^2 + (\omega + \omega^2)\alpha\beta = \alpha^2 + \beta^2 - \alpha\beta$$

$$= (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q$$

11. (c) We know that

$$\frac{1}{x-1} + \frac{1}{x-\omega} + \frac{1}{x-\omega^2} + \dots + \frac{1}{x-\omega^{n-1}} = \frac{n(x^{n-1})}{x^n - 1}$$

$$\text{Putting } x = 2, \text{ we get } \frac{1}{2-\omega} + \frac{1}{2-\omega^2} + \dots + \frac{1}{2-\omega^{n-1}} = \frac{n(2^{n-1})}{2^n - 1}$$

12. (c) We have, $1 + \omega + \omega^2 + \dots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega}$

$$\text{But } \omega^n = \cos\left(\frac{n\pi}{n}\right) + i \sin\left(\frac{n\pi}{n}\right) = \cos\pi + i \sin\pi = -1$$

$$\text{and } 1 - \omega = 2 \sin^2 \frac{\pi}{2n} - 2i \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}$$

$$= -2i \sin\left(\frac{\pi}{2n}\right) \left[\cos\frac{\pi}{2n} + i \sin\frac{\pi}{2n} \right]$$

$$\text{Thus, } 1 + \omega + \omega^2 + \dots + \omega^{n-1} = \frac{2[\cos(\pi/2n) - i \sin(\pi/2n)]}{-2i \sin(\pi/2n)}$$

$$= 1 + i \cot\left(\frac{\pi}{2n}\right)$$

13. (a) We have $|z_1| = |z_2| + |z_1 - z_2|$

$$\Rightarrow |z_1 - z_2|^2 = (|z_1| - |z_2|)^2 \Rightarrow |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = 0 \Rightarrow \frac{z_1}{z_2} \text{ is purely real}$$

$$\Rightarrow \operatorname{Im}\left(\frac{z_1}{z_2}\right) = 0$$

14. (b) Let $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$. Then

$$\left| \frac{z_1}{z_2} \right| = 1 \Rightarrow |z_1| = |z_2| \Rightarrow |z_1| = |z_2| = r_1$$

Now $\arg(z_1 z_2) = 0 \Rightarrow \arg(z_1) + \arg(z_2) = 0$

$$\Rightarrow \arg(z_2) = -\theta_1$$

Therefore,

$$z_2 = r_1(\cos(-\theta_1) + i \sin(-\theta_1)) = r_1(\cos\theta_1 - i \sin\theta_1) = \bar{z}_1$$

$$\Rightarrow \bar{z}_2 = \left(\overline{\overline{z}_1} \right) = z_1 \Rightarrow |z_2|^2 = z_1 z_2$$

15. (d) Let us denote the given determinant by Δ . Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix} = \begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix}$$

Clearly $\Delta = 0$ for $x = 0$.

16. (d) Origin (O) lies inside the circle

$$\text{Greatest value of } |z| = OC + r = \sqrt{13} + 4$$

$$\text{Least value of } |z| = r - OC = 4 - \sqrt{13}$$

$$\text{Required difference} = \sqrt{13} + 4 - 4 + \sqrt{13} = 2\sqrt{13}$$

17. (a) Let $\alpha = \frac{2 + 3i \sin\theta}{1 - 2i \sin\theta}$

$$\Rightarrow \alpha = \frac{(2 + 3i \sin\theta)(1 + 2i \sin\theta)}{(1 - 2i \sin\theta)(1 + 2i \sin\theta)} = \frac{(2 - 6\sin^2\theta) + i(7\sin\theta)}{1 + 4\sin^2\theta}$$

As α is purely imaginary, we have

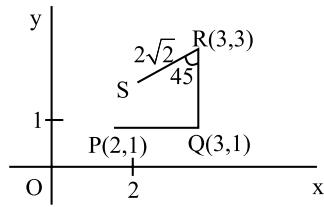
$$\operatorname{Re}(\alpha) = 0 \Rightarrow 2 = 6 \sin^2 \theta \text{ i.e. } \sin \theta = \pm \frac{1}{\sqrt{3}}$$

18. (a) By rotation theorem,

$$\frac{z - (3 + 3i)}{3 + i - (3 + 3i)} = \frac{2\sqrt{2}}{2} e^{(-\pi/4)i}$$

$$\frac{z - 3 - 3i}{-2i} = 1 - i$$

$$z - 3 - 3i = -2i - 1$$



$$z = 1 + i$$

$$19. (d) \bar{z} = \frac{1}{i-1}$$

$$\text{We have } z = \overline{(\bar{z})} \text{ giving } z = \frac{1}{\bar{i}-1} = \frac{1}{-i-1} = \frac{-1}{i+1}$$

$$20. (a) |z + 1| = |(z + 4) - 3| \leq |z + 4| + |3| \leq 3 + 3 = 6.$$

Note that $z = -7$ satisfies $|z + 4| = 3$ and $|z + 1| = 6$. So the maximum value of $|z + 1|$ is 6.

6. Quadratic Equations

- 1. (c)** The equation, $x^2 - x + a - 3 = 0$ must have atleast one negative root.

$$\text{For real roots, } D \geq 0 \Rightarrow 1 - 4(a - 3) \geq 0$$

$$\Rightarrow a \leq \frac{13}{4} \quad \Rightarrow a \in \left[3, \frac{13}{4} \right] \quad \dots (1)$$

Both root will be non-negative if :

$$D \geq 0, a - 3 \geq 0, 1 \geq 0$$

$$\Rightarrow a \leq \frac{13}{4}, a \geq 3 \quad \Rightarrow a \in \left[3, \frac{13}{4} \right] \quad \dots (2)$$

Thus equation will have atleast one negative root, if

$$a \in \left(-\infty, \frac{13}{4} \right] \cup \left[3, \frac{13}{4} \right] \text{ Using (1) and (2) } \Rightarrow a \in (-\infty, 3].$$

- 2. (b)** Here, $f(x) = x^2 + ax + b$, then

$$\begin{aligned} f(x+c) &= (x+c)^2 + a(x+c) + b \\ &= x^2 + (2c+a)x + c^2 + ac + b \end{aligned}$$

which show roots of $f(x)$ are transferred to $(x-c)$, i.e. roots of $f(x+c)=0$ are

$$c - c \text{ and } d - c.$$

Then, $x^2 + (2c+a)x + c^2 + ac + b^2 = 0$ has roots zero and $(d-c)$.

- 3. (c)** Here let S be the sum and P be the product of the roots,

$$\alpha^2 + \beta^2, \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\text{Now } S = (\alpha^2 + \beta^2) + \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \left(\frac{b^2 - 2ac}{a^2} \right) + \left(\frac{b^2 - 2ac}{c^2} \right)$$

$$= (b^2 - 2ac) \frac{(a^2 + c^2)}{a^2 c^2}$$

$$= (b^2 - 2ac) \frac{(a^2 + c^2)}{a^2 c^2}$$

$$\text{and the product } P = \frac{(\alpha^2 + \beta^2)^2}{\alpha^2 \beta^2} = \left(\frac{b^2 - 2ac}{a^2} \right)^2 \times \frac{1}{c^2}$$

Hence equation is $(acx)^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$

- 4. (a,b)** We can write the given equation as,

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

$$\text{or } p(x^2 - c^2) = 2(a+b)x^2 - 2c(a-b)x$$

$$\text{or } (2a+2b-p)x^2 - 2c(a-b)x + pc^2 = 0$$

For this equation to have equal roots

$$c^2(a-b)^2 - pc^2(2a+2b-p) = 0 \quad [\because c^2 \neq 0]$$

$$\Rightarrow (a-b)^2 - 2p(a+b) + p^2 = 0$$

$$\Rightarrow [p - (a+b)]^2 = (a+b)^2 - (a-b)^2 = 4ab$$

$$\Rightarrow p - (a+b) = \pm 2\sqrt{ab}$$

$$\Rightarrow p = a + b \pm 2\sqrt{ab} = \left(\sqrt{a} \pm \sqrt{b} \right)^2$$

$$\begin{aligned} \text{5. (a) Here } S &= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) \\ &= \alpha(\gamma + \delta) + \beta(\gamma + \delta) = (\alpha + \beta)(\gamma + \delta) = \frac{bm}{al} \end{aligned}$$

$$\begin{aligned} \text{Also } P &= (\alpha\gamma + \delta\beta)(\alpha\delta + \beta\gamma) \\ &= (\alpha^2 + \beta^2)\gamma\delta + \alpha\beta(\gamma^2 + \delta^2) \dots (\text{note}) \end{aligned}$$

$$= \frac{b^2 n l + m^2 a c - 4 a c n l}{a^2 l^2}$$

Hence from $x^2 - Sx + P = 0$, option (a) is correct.

$$\begin{aligned} \text{6. (a) Here } \alpha + \beta = -p \\ \gamma + \delta = -p \end{aligned} \Rightarrow \alpha + \beta = \gamma + \delta$$

$$\begin{aligned} \text{Now } (\alpha - \gamma)(\alpha - \delta) &= \alpha^2 - \alpha(\gamma + \delta) + \gamma\delta \\ &= \alpha^2 - \alpha(\alpha + \beta) + r = -\alpha\beta + r = -(-q) + r = q + r \end{aligned}$$

By symmetry of the results,

$$(\beta + \gamma)(\beta - \delta) = q + r, \text{ the ratio is 1.}$$

$$\begin{aligned} \text{7. (c) Here } \alpha + \beta = -p, \alpha\beta = 1 \\ \gamma + \delta = -q, \gamma\delta = 1 \end{aligned} \Rightarrow \alpha\beta + \gamma\delta \text{ (note)}$$

$$\text{Now } (\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$$

$$\begin{aligned} &= \{ \alpha\beta - \alpha(\alpha + \beta) + \gamma^2 \} \{ \alpha\beta + \delta(\alpha + \beta) + \delta^2 \} \\ &= \{ 1 + \gamma p + \gamma^2 \} \{ 1 - p\delta + \delta^2 \} \\ &= [(\gamma^2 + 1) + \gamma p] [(\delta^2 + 1) - p\delta] \\ &= (-q\gamma + \gamma p)(-q\delta - p\delta) = \gamma\delta(q^2 - p^2) \end{aligned}$$

$$\text{8. (d) Let } \frac{x^2 + 14x + 9}{x^2 + 2x + 3} = y$$

$$\Rightarrow x^2(1-y) + 2x(7-y) + 3(3-y) = 0$$

Hence $4(7-y)^2 - 12(1-y)(3-y) \geq 0$ gives

$$-2y^2 - 2y + 40 \geq 0$$

$$\Rightarrow y^2 + y - 20 \leq 0 \Rightarrow (y+5)(y-4) \leq 0$$

$$\therefore -5 \leq y \leq 4$$

Note : Theory of maxima minima may be used to find the extreme values in the above example.

- 9. (c)** Let $x - \alpha$ be the common factor

then $x = \alpha$ is root of the corresponding equations

$$\therefore \alpha^2 - 11\alpha + a = 0 \text{ and } \alpha^2 - 14\alpha + 2a = 0$$

$$\text{Subtracting } 3\alpha - a = 0 \Rightarrow \alpha = \frac{a}{3}$$

$$\text{Hence } \frac{a^2}{9} - 11\frac{a}{3} + a = 0 \Rightarrow a = 0 \text{ or } a = 24$$

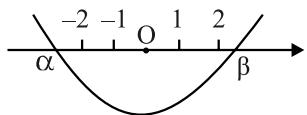
$$\text{Since } a \neq 0, a = 24$$

\therefore the common factor of $\begin{cases} x^2 - 11x + 24 \\ x^2 - 14x + 48 \end{cases}$ is clearly $x - 8$

- 10. (d)** Since the equation has two distinct roots α and β , the discriminant $b^2 - 4ac > 0$. We must have

$$f(x) = ax^2 + bx + c < 0 \text{ for } \alpha < x < \beta$$

Since $\alpha < 0 < \beta$, we must have $f(0) = c < 0$
and $f(1) = a + b + c < 0$ i.e., $a + |b| + c < 0$.



Next, since $\alpha < -2$, $2 < \beta$,

$$f(-2) = 4a - 2b + c < 0$$

$$\text{and } f(2) = 4a + 2b + c < 0, \text{ i.e., } 4a + 2|b| + c < 0.$$

11. (d) Since, $\sin^{-1} \theta$ is defined only when, $-1 \leq \theta \leq 1$

$$\Rightarrow -1 \leq x^2 - 4x + 5 \leq 1$$

$$\Rightarrow x^2 - 4x + 4 \leq 0 \text{ and } x^2 - 4x + 6 \geq 0$$

$$\Rightarrow (x-2)^2 \leq 0 \text{ and } (x-2)^2 + 2 \geq 0 \Rightarrow x=2 \text{ is only solution}$$

$$\text{Putting } x=2, \text{ we get } 4 + 2a + \frac{\pi}{4} = 0 \Rightarrow a = -2 - \frac{\pi}{4}$$

$$12. (a) \text{ Here, } \frac{(16)^{\frac{4}{x}}}{(2^{x+3})} > 1$$

$$\Rightarrow \frac{2^{\frac{4}{x}}}{2^{x+3}} > 1 \text{ or } 2^{\frac{4}{x}-x-3} > 1 \text{ i.e. } 2^{\frac{4}{x}-x-3} > 2^0$$

$$\Rightarrow \frac{4}{x} - x - 3 > 0 \Rightarrow \frac{(x^2 + 3x - 4)}{x} > 0 \text{ or}$$

$$\frac{-(x+4)(x-4)}{x} > 0$$

$$\text{Using number line rule, } \begin{array}{ccccccc} & + & & - & + & & - \\ \hline & & & -4 & 0 & 1 & \end{array}$$

$$\Rightarrow x \in (-\infty, -4) \cup (0, 1)$$

13. (d) $|x| = x \forall x \geq 0$ and $|x| = -x \forall x < 0$

Case 1: $x \geq 0$

$$x^2 + 5x + 6 = (x+2)(x+3) = 0 \Rightarrow x = -2, -3 \text{ but we already assumed } x \geq 0 \text{ so contradiction.}$$

Case 2: $x < 0$

$$x^2 - 5x + 6 = (x-2)(x-3) = 0 \Rightarrow x = 2, 3 \text{ but we already assumed } x < 0 \text{ so contradiction.}$$

Thus real roots does not exists.

$$14. (b) x = 2 + 2^{\frac{1}{3}} + 2^{\frac{2}{3}}$$

$$\Rightarrow x - 2 = 2^{\frac{1}{3}} + 2^{\frac{2}{3}} = 2^{\frac{1}{3}} \left(2^{\frac{1}{3}} + 1 \right)$$

Now cubing on both sides we get

$$(x-2)^3 = \left(2^{\frac{1}{3}} \right)^3 \left(2^{\frac{1}{3}} + 1 \right)^3$$

$$\Rightarrow x^3 - 8 + 12x - 6x^2 = 2 \left(2 + 1 + 3 \left(2^{\frac{1}{3}} + 2^{\frac{2}{3}} \right) \right)$$

$$\Rightarrow x^3 - 8 + 12x - 6x^2 = 2(3 + 3(x-2))$$

$$\Rightarrow x^3 - 8 + 12x - 6x^2 = 6 + 6x - 12$$

$$\Rightarrow x^3 - 6x^2 + 6x - 2 = 0$$

$$\Rightarrow x^3 - 6x^2 + 6x = 2$$

15. (d) Let the two roots be a and b

$$a + b = 2 \quad \dots\dots(1)$$

$$a^3 + b^3 = 98$$

$$(a+b)(a^2 - ab + b^2) = 98 \quad (\text{from (1)})$$

$$2(a^2 + 2ab + b^2 - 3ab) = 98$$

$$\left[(a+b)^2 - 3ab \right] = \frac{98}{2}$$

$$[2^2 - 3ab] = 49 \quad (\text{from (1)})$$

$$[4 - 3ab] = 49$$

$$4 - 49 = 3ab$$

$$3ab = -45$$

$$ab = -15 \quad \dots\dots(2)$$

Now, Equation with roots a and b can be written as: $(x-a)(x-b)$

$$\Rightarrow (x^2 - bx - ax + ab)$$

$$\Rightarrow (x^2 - (a+b)x + ab)$$

$$\Rightarrow (x^2 - 2x - 15) \quad (\text{from (2)})$$

16. (d) $x^2 - x + 1 = 0$ has its roots $-\omega, -\omega^2$.

$$\text{Now } (-\omega)^{101} + (-\omega^2)^{107} = -\{\omega^2 + \omega^4\} = -(\omega^2 + \omega) = 1$$

$$17. (c) 2^{(x-1)(x^2+5x-50)} = 1 = 2^0 \therefore (x-1)(x^2+5x-50) = 0$$

$$\Rightarrow (x-1)(x+10)(x-5) = 0 \Rightarrow x = 1, 5, -10$$

$$\therefore \text{Required sum} = 1 + 5 - 10 = -4.$$

18. (a) As $1-p$ is root of $x^2 + px + 1 - p = 0$

$$\Rightarrow (1-p)^2 + p(1-p) + (1-p) = 0$$

$$(1-p)[1-p+p+1] = 0 \Rightarrow p = 1$$

$$\therefore \text{Given equation becomes } x^2 + x = 0 \Rightarrow x = 0, -1$$

19. (c) As $x^2 + px + q = 0$ has equal roots $\therefore p^2 = 4q$ and one root of $x^2 + px + 12 = 0$ is 4. So, the other root must be 3.

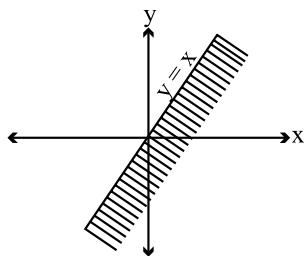
$$\therefore p = 3 + 4 = 7$$

$$\Rightarrow q = \frac{p^2}{4} = \frac{49}{4}$$

20. (c) Since, $b^2 - 4ac = 1 - 36 = -35 < 0$ so real roots do not exist.

7. Linear Inequalities

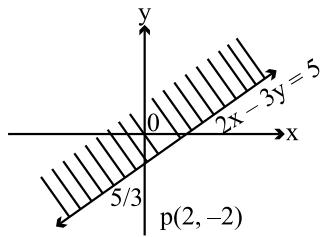
1. (d) $y - x \leq 0$



Shaded region represents the inequality.

2. (a) $2x - 3y < 5$

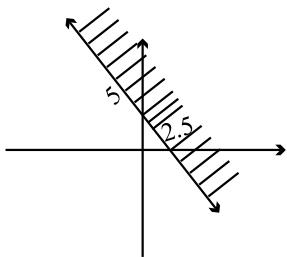
Points → 0(0,0) & P(2, -2)



Shaded region satisfy Inequality

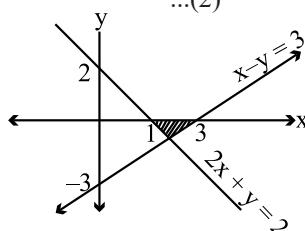
So, '0' (0,0) is inside & P(2, -2) is outside.

3. (b) $2x + y > 5$



4. (b) $2x + y \geq 2$... (1)

$x - y \leq 3$... (2)



By observing options, only one option (B) lies in shaded area.

or

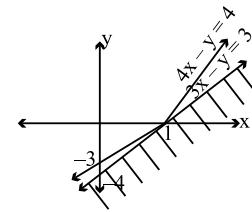
It can be solved analytically as well. by assuming '2' equations as—

$2x + y = 2$... (i)

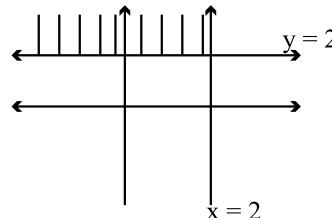
$x - y = 3$... (ii)

5. (a) $3x - y > 3$... (i)
 $4x - y > 4$... (ii)

There is no option correct. The in equalities will have solution for $x > 1$ & All values of y :



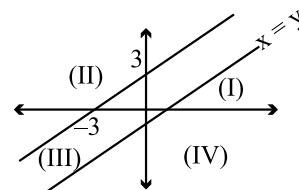
6. (a) Graph of $x \leq 2$ & $y \geq 2$



7. (b) In 3rd quadrant, $x < 0$ & $y < 0$.

8. (c) $y > x + 3$... (i)

$x \leq y$... (ii)



9. (d) Form the given graph, It is clear that

$y < 250$ (i)

$x < 250$ (ii)

$2x + y < 600$ (iii)

So, option D is correct.

10. (d) $|x - 1| < 0$

It has No Sol, because $|x|$ is always greater than '0'.

\Rightarrow s

11. (c) $x^2 + x + |x| + 1 \leq 0$... (i)

For $x \geq 0$ $x^2 + x + 1 \leq 0$

$$x^2 + 2x + 1 \leq 0$$

$$(x + 1)^2 \leq 0 \quad \dots(\text{i})$$

So, No solution of x equation of ... (ii)

For $x < 0$ $x^2 + x - x + 1 \leq 0$

$$x^2 + 1 \leq 0 \quad \dots(\text{iii})$$

have also No sol of ' x '.

So Ans will be (c).

12. (a) $|x + 3| > |2x - 1|$

Case (1) $x + 3 > |2x - 1|$ for $x > -3$

(a) $x + 3 > 2x - 1$ for $x > -3$ and $x > 1/2$

i.e.. for $x > 1/2$

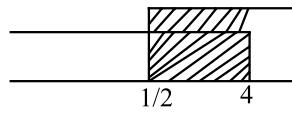
$$4 > x$$

$$x \in s \left(\frac{1}{2}, 4 \right)$$

(b) $x + 3 > -(2x - 1)$ for $x > -3$

and

$$\Rightarrow x + 3 > -2x + 1 \quad x < 1/2$$



$$\begin{aligned}\Rightarrow 3x > -2 \\ \Rightarrow x > -2/3 \\ x \in \left(-\frac{2}{3}, \frac{1}{2}\right)\end{aligned}$$

... (ii)

from (i) & (ii)

$$\begin{array}{c} \text{Number line: } -2/3, -2/3, -2/3 \\ x \in \left(-\frac{2}{3}, 4\right) \end{array}$$

13. (c) $(x+3)^5 - (x-1)^5 \geq 244$

$$\begin{aligned}\text{Let } y = \frac{(x+3)+(x-1)}{2} \\ \Rightarrow y = x + 1 \\ \therefore (y+2)^5 - (y-2)^5 \geq 244 \\ \Rightarrow 2\{{}^5C_1 y^4 \cdot 2 + {}^5C_3 y^2 \cdot 2^3 + {}^5C_5 \cdot 2^5\} \geq 244 \\ \Rightarrow 2\{10y^4 + 80y^2 + 32\} \geq 244 \\ \Rightarrow 4\{5y^4 + 40y^2 + 16\} \geq 244 \\ \Rightarrow 4y^4 + 40y^2 + 16 \geq 61 \\ \Rightarrow y^4 + 8y^2 - 9 \geq 0 \\ \Rightarrow (y^2 + 9)(y^2 - 1) \geq 0 \Rightarrow y^2 \geq 1 \\ \text{i.e., } y \leq -1 \text{ or } y \geq 1\end{aligned}$$

$$\Rightarrow x + 1 \leq -1 \text{ or } x + 1 \geq 1$$

$$\Rightarrow x \in (-\infty, -2] \cup [0, \infty)$$

14. (a) $||x-2|-1| \geq 3$

$$\begin{aligned}\Rightarrow |x-2|-1 \leq -3 \text{ or } |x-2|-1 \geq 3 \\ \Rightarrow |x-2| \leq -2 \text{ or } |x-2| \geq 4 \\ \Rightarrow \text{No solution or } x-2 \leq -4 \text{ or } x-2 \geq 4 \\ \Rightarrow x \leq -2 \text{ or } x \geq 6 \Rightarrow x \in (-\infty, -2] \cup [6, \infty)\end{aligned}$$

15. (d) $1 \leq \frac{3x^2 - 7x + 8}{x^2 + 1} \leq 2$

$$\Rightarrow 2x^2 - 7x + 7 \geq 0 \text{ and } x^2 - 7x + 6 \leq 0$$

$$\Rightarrow x \in \mathbb{R} \text{ and } x \in [1, 6]$$

$$\therefore x \in [1, 6]$$

16. (b) $f(x) > g(x) \Rightarrow \frac{2x}{2x^2 + 5x + 2} > \frac{1}{x+1}$

$$\Rightarrow \frac{2x}{(2x+1)(x+2)} - \frac{1}{(x+1)} > 0$$

$$\Rightarrow \frac{2x^2 + 2x - (2x^2 + 5x + 2)}{(2x+1)(x+2)(x+1)} > 0$$

$$\Rightarrow \frac{-(3x+2)}{(2x+1)(x+2)(x+1)} > 0$$

$$\begin{array}{ccccccc} < - & + & - & + & - & - & \\ \leftarrow & \overbrace{\hspace{1cm}}_{-2} & \overbrace{\hspace{1cm}}_{-1} & \overbrace{\hspace{1cm}}_{-2/3} & \overbrace{\hspace{1cm}}_{-1/2} & \rightarrow & \end{array}$$

$$\Rightarrow x \in (-2, -1) \cup (-2/3, -1/2)$$

17. (d) $|x^2 + 3x| + x^2 - 2 \geq 0 \quad \dots(\text{i})$

case (1) $x^2 + 3x \geq 0$

$x(x+3) \geq 0 \quad \dots(\text{A})$

equation (1) becomes

$$x^2 + 3x + x^2 - 2 \geq 0$$

$$2x^2 + 3x - 2 \geq 0$$

$$\Rightarrow 2x^2 + 4x - x - 2 \geq 0$$

$$2x(x+2) - (x+2) \geq 0$$

$$(x+2)(2x-1) \geq 0$$



$$x \leq -2 \text{ or } x \geq 1/2 \quad \dots(\text{B})$$

from (A) & (B),

$$x \leq -2 \text{ or } x \geq 1/2$$

case (2), $x^2 + 3x \leq 0$

$$\Rightarrow x \in (-3, 0) \quad \dots(\text{C})$$

And, equation (1) becomes,

$$-x^2 - 3x + x^2 - 2 \geq 0$$

$$-3x - 2 \geq 0$$

$$-(3x+2) \geq 0$$

$$(3x+2) \leq 0$$

$$\Rightarrow x < -2/3 \quad \dots(\text{D})$$

from (C) & (D) $x \in (-3, -2/3)$

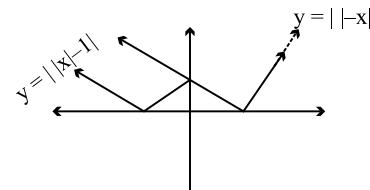
from both cases,

$$x \in \left[-3, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, \infty\right]$$

(d) given Ans is in correct.

18. (d) $||x|-1| < |1-x|$

Graph of $y = ||x|-1|$ & $y = |1-x|$



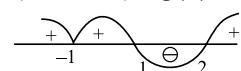
$$(-\infty, 0)$$

20. (a) $|x|^{x^2-x-2} < 1$

taking log both side,

$$(x^2 - x - 2) \log|x| < \log 1$$

$$(x^2 - x - 2) \log|x| < 0$$



$$\log|x| = 0$$

$$\text{at } x = 1, -1$$

$$x^2 - x - 2 = 0$$

$$x(x-2) + (x-2) = 0$$

$$x = -1, 2$$

$$(1, 2)$$

8. Permutations and Combinations

- 1. (d)** Required number of possible outcomes

= Total number of possible outcomes – Number of possible outcomes in which 5 does not appear on any dice = $6^3 - 5^3 = 91$.

- 2. (c)** We can arrange n white and n black balls alternately in the following ways

(i) W B W B . . . (ii) B W B W . . .

So required number of ways = $n! \times n! + n! \times n! = 2(n!)^2$

- 3. (b)** The number of ways to choose 4 novels out of 6 is 6C_4 . The number of ways to choose 1 dictionary out of 3 is 3C_1 . As the place of dictionary is fixed, so total number of ways = ${}^6C_4 \cdot {}^3C_1 \cdot 4! = 15 \cdot 3 \cdot 24 = 1080$

- 4. (b)** We have : $30 = 2 \times 3 \times 5$. So, 2 can be assigned to either a or b or c i.e. 2 can be assigned in 3 ways. Similarly, each of 3 and 5 can be assigned in 3 ways. Thus, the number of solutions is $3 \times 3 \times 3 = 27$.

- 5. (b)** Let the number of green balls be x . Then the number of red balls is $2x$. Let the number of blue balls be y . Then,

$$x + 2x + y = 10 \Rightarrow 3x + y = 10 \Rightarrow y = 10 - 3x$$

Clearly, x can take values 0, 1, 2, 3. The corresponding values of y are 10, 7, 4 and 1. Thus, the possibilities are $(0, 10, 0), (2, 7, 1), (4, 4, 2)$ and $(6, 1, 3)$, where (r, b, g) denotes the number of red, blue and green balls.

- 6. (a)** There are three ways to make four letter words from the letters of the word BARRACK.

- (i) When two letters are same i.e. A, A, R, R

$$\text{So, number of words} = \frac{{}^2C_2 \cdot 4!}{2!2!} = 6$$

- (ii) When one letter is same and other is different

$$\text{So, number of words} = \frac{{}^2C_1 \times {}^4C_2 \times 4!}{2!} = 144$$

- (iii) When all letters are different So, number of words = ${}^5C_4 \times 4! = 120$

$$\therefore \text{Total number of words} = 6 + 144 + 120 = 270$$

- 7. (c)** Six consonants and three vowels can be selected from 10 consonants and 4 vowels in ${}^{10}C_6 \times {}^4C_3$ ways. Now, these 9 letters can be arranged in $9!$ ways. So, required number of words = ${}^{10}C_6 \times {}^4C_3 \times 9!$

- 8. (a)** Any interior intersection point corresponds to 4 of the fixed points, namely the 4 end points of the intersecting segments. Conversely, any 4 labeled points determine a quadrilateral, the diagonals of which intersect once within the circle .

$$\text{Number of interior intersection points} = {}^9C_4 = 126.$$

- 9. (d)** The number of words in all formed by using the letters of the word SMALL = $\frac{5!}{2!} = 60$

Let's count backwards.

The 59th word is SMALL ∴ 58th word is SMALL

- 10. (d)** Number of ways = Arrangement of $(m - 1)$ things of one kind and $(n - 1)$ things of the other kind = $\frac{(m+n-2)!}{(m-1)!(n-1)!}$

- 11. (c)** For a particular class total number of different tickets from first intermediate station = 5

Similarly, number of different tickets from second intermediate station = 4

So total number of different tickets = $5 + 4 + 3 + 2 + 1 = 15$

And same number of tickets for another class

⇒ total number of different tickets = 30 and number of selection = ${}^{30}C_{10}$

- 12. (c)** We have, $\frac{{}^{n+2}C_6}{{}^{n-2}P_2} = 11 \Rightarrow \frac{(n+2)(n+1)n(n-1)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \Rightarrow (n+2)(n+1)n(n-1) = 11 \cdot 10 \cdot 9 \cdot 8 \Rightarrow n = 9$

- 13. (c)** Total number of hand shales = ${}^{20}C_2$ of those no indian female shakes hand with male

$$\Rightarrow 5 \times 10 = 50 \text{ hand shales}$$

No American wife shakes hand with her husband = $5 \times 1 = 5$ hand shales

⇒ total number of hand shales occurred = ${}^{20}C_2 - (50 + 5) = 190 - 55 = 135$

- 14. (d)** Number of ways of arranging n things in circle = $(n - 1)!$

Since r are alike so that arrangements = $\frac{(n - 1)!}{r!}$

- 15. (d)** When A has B or C to his right we have AB or AC.

When B has C or D to his right we have BC or BD.

Thus we must have ABC or ABD or AC and BD.

For ABC D, E, F on a circle, number of ways = $3! = 6$

For ABD C, E, F on a circle, number of ways = $3! = 6$

For AC, BD E, F the number of ways = $3! = 6$

$$\Rightarrow \text{Total} = 18$$

- 16. (d)** Number of ways = $({}^3C_2) \times ({}^9C_2) = 3 \times \frac{9 \times 8}{2} = 108$

- 17. (a)** G₁ B₁ G₂ B₂ G₃ B₃ G₄ B₄ G₅ B₅ G₆ B₆

$$\text{Number of ways} = 6! \times 6!$$

Now if boys sits first then the number of arrangement = $6! \times 6!$

$$\text{Total number of ways} = 2 \cdot 6! \cdot 6!$$

- 18. (a)** Time required = $\frac{15 \times 15 \times 15 - 1}{2} \times \frac{10}{60 \times 60} = \frac{1687}{360}$ hrs.

$$\approx 4 \text{ hrs. } 41 \text{ min. } 10 \text{ seconds} > 4\frac{1}{2} \text{ hrs.}$$

19. (a) sum = $2700(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 810(1 + 4 + 9) + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)27$
= $2700(45) + 810(14) + 27(45)$
= 134055

20. (b) The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must

be formed. In the time-table for a week the number 1 is repeated twice, the number 2 is repeated 3 times and the number 3 is repeated twice. The number of different time-tables is equal to

$$P(2,3,2) = \frac{7!}{2! 3! 2!} = 210$$

9. Binomial Theorem and its simple Applications

1. (c) We know that ${}^n C_r |_{\max i} = \begin{cases} {}^n C_{n/2}, & n = \text{even} \\ {}^n C_{\frac{n-1}{2}}, & n = \text{odd} \end{cases}$

$${}^n C_r |_{\max i} = {}^{20} C_{10}$$

2. (a) The $(r+1)^{\text{th}}$ term in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$ is given by

$$\begin{aligned} T_{r+1} &= {}^{10} C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r = {}^{10} C_r \frac{x^{5-(r/2)}}{3^{5-(r/2)} \cdot 2^r x^{2r}} \\ &= {}^{10} C_r \frac{3^{(3r/2)-5}}{2^r} x^{5-(5r/2)} \end{aligned}$$

For T_{r+1} to be independent of x , we must have $5 - \left(\frac{5r}{2}\right) = 0$ or $r = 2$. Thus, the 3rd term is independent of x and is equal to

$${}^{10} C_2 \frac{3^{3-5}}{2^2} = \frac{10 \times 9}{2} \times \frac{3^{-2}}{4} = \frac{5}{4}.$$

3. (a) We have, $(a+b)^5 + (a-b)^5 = 2 \{a^5 + {}^5 C_2 a^3 b^2 + {}^5 C_4 a b^4\}$

$$\text{With } a=x, b=\sqrt{x^3-1} \quad \therefore (x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$$

$$= 2 \{x^5 + 10x^3(x^3-1) + 5x(x^3-1)^2\}$$

= Sum of the coeff. of odd degree terms is

$$2\{1-10+5+5\} = 2$$

$$\mathbf{4. (b)} \sum_{r=0}^{2n} (-1)^r \cdot ({}^{2n} C_r)^2 = \sum_{r=0}^{2n} (-1)^r \cdot {}^{2n} C_r \cdot {}^{2n} C_r$$

$$= \sum_{r=0}^{2n} (-1)^r \cdot {}^{2n} C_r \cdot {}^{2n} C_{2n-r} = \text{coefficient of } x^{2n} \text{ in } \frac{(1-x)^{2n}}{(1+x)^{2n}}$$

$$= \text{coefficient of } x^{2n} \text{ in } (1-x^2)^{2n} = (-1)^n \cdot {}^{2n} C_n.$$

5. (b) Clearly $a_r = {}^n C_r$

$$\Rightarrow \frac{a_r}{a_{r-1}} = \frac{{}^n C_r}{{}^n C_{r-1}} = \frac{(n-r+1)}{r}$$

$$\Rightarrow 1 + \frac{a_r}{a_{r-1}} = \frac{n+1}{r}$$

$$\Rightarrow \prod_{r=1}^n \left(1 + \frac{a_r}{a_{r-1}}\right) = \prod_{r=1}^n \frac{(n+1)}{r} = \frac{(n+1)^n}{n!}$$

$$\mathbf{6. (b)} (1+x)^2 (1+x^2)^3 (1+x^3)^4 = (1+x^2+2x) \times (1+x^6+3x^2+3x^4) \times (1+4x^3+6x^6+4x^9+x^{12})$$

$$\text{So, coefficient of } x^{10} + 36 + 8 + 8 = 52$$

$$\mathbf{7. (c)} 3^{2003} = 3^{2001} \cdot 3^2 = 9(27)^{667} = 9(28-1)^{667}$$

$$= 9({}^{667} C_0 (28)^{667} - {}^{667} C_1 (28)^{666} + \dots + {}^{667} C_{667} (-1)^{667})$$

that means if we divide 3^{2003} by 28, remainder is 19.

$$\text{Thus, } \left\{ \frac{3^{2003}}{28} \right\} = \frac{19}{28}.$$

$$\mathbf{8. (d)} \text{ Let } f' = (5-2\sqrt{6})^n$$

$$\text{So, } 1+f+f' = (5+2\sqrt{6})^n + (5-2\sqrt{6})^n = \text{an integer} \\ \Rightarrow f+f'=1 \quad \dots (i)$$

$$\text{Now, } (1+f) \times f' = (5+2\sqrt{6})^n \times (5-2\sqrt{6})^n = 1$$

$$\therefore I = \frac{1}{f'} - f = \frac{1}{1-f} - f$$

$$\mathbf{9. (b)} T_{r+1} = {}^{18} C_r \left(x^{\frac{1}{3}}\right)^{18-r} \left(\frac{1}{2x^{\frac{1}{3}}}\right) = {}^{18} C_r x^{\frac{6-2r}{3}} \frac{1}{2^r}$$

$$\text{Put } 6 - \frac{2r}{3} = -2 \Rightarrow r = 12 \text{ and } 6 - \frac{2r}{3} = -4 \Rightarrow r = 15$$

$$\therefore \frac{\text{Coefficient of } x^{-2}}{{\text{Coefficient of } x^{-4}}} = \frac{{}^{18} C_{12} \frac{1}{2^{12}}}{{}^{18} C_{15} \frac{1}{2^{15}}} = 182$$

10. (c) We have

$$(1+x)^n = \sum_{r=0}^n C_r x^r \quad \dots (i)$$

$$(1-x)^n = \sum_{r=0}^n (-1)^r C_r x^r \quad \dots (ii)$$

Adding equations (i) and (ii) we have,

$$(1+x)^n + (1-x)^n = C_0 + C_2 x^2 + C_4 x^4 + \dots$$

Integrating both sides with respect to x from 0 to 1, we get

$$\frac{2^{n+1}}{n+1} = \frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \dots$$

11. (b) We have

$$\begin{aligned} &(1+x+x^2+x^3+x^4)^n (x-1)^{n+3} \\ &= \left(\frac{1-x^5}{1-x}\right)^n (1-x)^{n+3} = (1-x^5)^n (1-x)^3 \end{aligned}$$

$$= (-x^3 + 3x^2 - 3x + 1) \sum_{r=0}^n {}^n C_r (-1)^r x^{5r}$$

$$= \sum_{r=0}^n {}^n C_r (-1)^r x^{5r+3} + 3 \sum_{r=0}^n {}^n C_r (-1)^r x^{5r+2}$$

$$= -3 \sum_{r=0}^n {}^n C_r (-1)^r x^{5r+1} + 3 \sum_{r=0}^n {}^n C_r (-1)^r x^{5r}$$

For term containing x^{83} , we have

$$5r+3=83 \Rightarrow r=16$$

whereas $5r+2=83$, $5r+1=83$ and $5r=83$ give no integral value of r .

Hence there is only one term containing x^{83} whose coefficient is $-{}^n C_{16}$.

$$\mathbf{12. (a)} (17)^{256} = (289)^{128}$$

$$= (300-11)^{128}$$

$$= {}^{128} C_0 (-11)^{128} + 100m, \text{ for some integer } m$$

$$\begin{aligned}
&= 11^{128} + 100m \\
&= (10+1)^{128} + 100m \\
&= {}^{128}C_0 1^{128} + {}^{128}C_1 10 + 100 m_1 + 100m \text{ for some integer } m_1 \\
&= 1 + 1280 + 100k, m + m_1 = k \\
&= 1281 + 100k
\end{aligned}$$

Hence the required number is 81.

13. (c) We have

$$\begin{aligned}
&\text{Coefficient of } x^4 \text{ in } (1+x+x^2+x^3)^{11} \\
&= \text{coefficient of } x^4 \text{ in } (1+x^2)^{11} (1+x)^{11} \\
&= \text{coefficient of } x^4 \text{ in } (1+x)^{11} + \text{coefficient of } x^2 \text{ in } 11 \cdot (1+x)^{11} + \text{constant term is} \\
&\quad {}^{11}C_2 \cdot (1+x)^{11} \\
&= {}^{11}C_4 + 11 \cdot {}^{11}C_2 + {}^{11}C_2 \\
&= 990.
\end{aligned}$$

$$14. \text{ (b)} \frac{(n+1)(n+2)}{2} = 45$$

$$\begin{aligned}
&\Rightarrow n^2 + 3n - 88 = 0 \\
&\Rightarrow n = 8
\end{aligned}$$

$$15. \text{ (b)} {}^nC_4 a^{n-4} (-b)^4 = -\left({}^nC_5 n^{n-5} (-b)^5 \right) \Rightarrow \frac{a}{b} = \frac{n-4}{5}$$

$$16. \text{ (a)} (a+x)^n = 1 + 8x + 24x^2 + \dots$$

By using Binomial $(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \dots$

So

$$\begin{aligned}
na &= 8 \\
n^2a^2 &= 64 \\
\text{And } \frac{n(n-1)}{2} a^2 &= 24
\end{aligned}$$

So

$$\begin{aligned}
\frac{n(n-1)}{2} \left(\frac{64}{n^2} \right) &= 24 \Rightarrow \frac{2n}{n-1} = \frac{8}{3} \Rightarrow 6n = 8n - 8 \Rightarrow n = 4 \\
\Rightarrow na &= 8 \Rightarrow a = 2 \text{ and } n = 4
\end{aligned}$$

17. (a) Perform this with modular arithmetic

$$\begin{aligned}
7^1 &\equiv 7 \pmod{100} \\
7^2 &\equiv 49 \pmod{100} \\
7^3 &\equiv 43 \pmod{100} \\
7^4 &\equiv 1 \pmod{100}
\end{aligned}$$

But,

$$1995 = 4 \times 498 + 3$$

Therefore,

$$7^{1995} \equiv 7^{4 \cdot 498 + 3} \equiv (7^4)^{498} \cdot 7^3 \equiv 1^{484} \cdot 43 \equiv 43 \pmod{100}$$

So, the remainder is 43.

18. (d)

19. (d)

$$20. \text{ (b)} (1+x)(1-x)^n = (1-x)^n + x(1-x)^n$$

$$\therefore \text{ Coefficient of } x^n \text{ is } = (-1)^n + (-1)^{n-1} {}^nC_1 = (-1)^n (1-n)$$

10. Sequences and Series

1. (c) We have $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = \lambda$ (say)

$$\Rightarrow p = \frac{a-x}{\lambda x}, q = \frac{a-y}{\lambda y}, r = \frac{a-z}{\lambda z}$$

Now p, q, r are in A.P.

$$\Rightarrow \frac{a-x}{\lambda x}, \frac{a-y}{\lambda y}, \frac{a-z}{\lambda z} \text{ are in A.P.}$$

$$\Rightarrow \frac{a-x}{x}, \frac{a-y}{y}, \frac{a-z}{z} \text{ are in A.P.}$$

$$\Rightarrow \frac{a}{x} - 1, \frac{a}{y} - 1, \frac{a}{z} - 1 \text{ are in A.P.}$$

$$\Rightarrow \frac{a}{x}, \frac{a}{y}, \frac{a}{z} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \text{ are in A.P.}$$

$\Rightarrow x, y, z$ are in HP.

2. (b)

$$3. (d) 1 + \frac{3}{2} + \frac{7}{4} + \frac{15}{8} + \frac{31}{16} + \dots$$

$$= (2-1) + \left(2 - \frac{1}{2}\right) + \left(2 - \frac{1}{4}\right) + \left(2 - \frac{1}{8}\right) + \dots + 20 \text{ terms}$$

$$= (2+2+\dots+20 \text{ terms}) - \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + 20 \text{ terms}\right)$$

$$= 2 \times 20 - \left(\frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} \right) = 40 - \left(\frac{1 - \left(\frac{1}{2}\right)^{20}}{\frac{1}{2}} \right)$$

$$= 40 - 2 + 2 \left(\frac{1}{2}\right)^{20} = 38 + \frac{1}{2^{19}}$$

4. (d) The first sequence is, $1 + 11 + 21 + 31 + 41 + 51 + 61 + 71 + 81 + 91 + \dots$ 100 term

last term will be $= 1 + (100-1) \times 10 = 991$

and for the sequence $31 + 36 + 41 + 46 + 51 + \dots$ 100 term
last term will be

$$= 31 + (100-1) \times 5 = 526$$

Now we have the common term as, $31 + 51 + 71 + \dots$

and n^{th} term of this sequence will be $= 31 + (n-1) \times 20 = 20n + 11$

and this will be less than 526

So, $20n + 11 < 526$

$20n < 515$

$$n < \frac{515}{20} = 25.75$$

So, $n = 25$

Hence the largest common term $= 20 \times 25 + 11 = 511$.

$$5. (b) S_n = \frac{n}{2} [2a + (n-1)d]$$

Here, $a = 1, d = 2$ and $S_n = 1357$

$$\Rightarrow 1357 = \frac{n}{2} [2 + 2(n-1)]$$

$$\Rightarrow 1357 = n^2$$

$$\Rightarrow n = 37$$

$$6. (a) \frac{t_4}{t_6} = \frac{ar^3}{ar^5} = \frac{1}{4}$$

$$\Rightarrow r = 2$$

$$t_2 + t_5 = 216$$

$$\Rightarrow ar + ar^4 = 216$$

$$\Rightarrow 18a = 216$$

$$\Rightarrow a = 12$$

$$\therefore t_1 = a = 12$$

7. (c) Arithmetic mean of a and $b = \frac{a+b}{2}$ and geometric mean of a and $b = \sqrt{ab}$

$$\text{According to question, } \frac{a+b}{2} = 5\sqrt{ab} \Rightarrow (a+b)^2 = 100ab$$

$$\text{Now, } (a+b)^2 - (a-b)^2 = 4ab$$

$$100ab - (a-b)^2 = 4ab \Rightarrow 96ab = (a-b)^2$$

$$\therefore \frac{(a+b)^2}{(a-b)^2} = \frac{100}{96} \Rightarrow \frac{a+b}{a-b} = \frac{10}{4\sqrt{6}}$$

$$\Rightarrow \frac{a+b}{a-b} = \frac{5}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12}$$

8. (b) Since $x, 2y, 3z$ are in AP, this implies that,

$$2y - x = 3z - 2y$$

$$2y + 2y = x + 3z$$

$$4y = x + 3z \quad \dots\dots(1)$$

Also, it is given that x, y, z are in GP which implies that,

$$\frac{y}{x} = \frac{z}{y} = r, \text{ Here } r \text{ is common ratio of GP}$$

$$y^2 = xz \quad \dots\dots(2)$$

Take square on both sides of (1)

$$(4y)^2 = (x + 3z)^2$$

$$16y^2 = x^2 + 9z^2 + 6xz$$

$$16xz = x^2 + 9z^2 + 6xz \quad (\text{Since } y^2 = xz \text{ from (2)})$$

$$0 = x^2 + 9z^2 + 6z - 16xz$$

$$0 = x^2 - 10xz + 9z^2$$

This gives,

$$x^2 - 10xz + 9z^2 = 0$$

$$x^2 - 9xz - xz + 9z^2 = 0$$

$$x(x-9z) - z(x-9z) = 0$$

$$(x - 9z)(x - z) = 0$$

Note that $x \neq z \Rightarrow x - z \neq 0$

This gives,

$$x - 9z = 0$$

$$x = 9z$$

From (1), we have

$$4y = 9z + 3z$$

$$4y = 12z$$

$$y = 3z$$

Thus the common ratio is given by,

$$r = \frac{y}{x} = \frac{3z}{9z} = \frac{1}{3}$$

9. (b) a, x, b are in AP

$$\Rightarrow x = \frac{a+b}{2} \quad \dots\dots(1)$$

a, y, b are in GP

$$\Rightarrow y^2 = ab \quad \dots\dots(2)$$

a, z, b are in HP

$$\Rightarrow 2z = \frac{2ab}{a+b}$$

$$\Rightarrow z = \frac{ab}{x} \quad \text{from (1)}$$

Also $x = 9z$

$$\Rightarrow z = \frac{ab}{9z}$$

$$\Rightarrow 9z^2 = y^2 \quad \text{from (2)}$$

$$\therefore 3z = y$$

$$\therefore x = 3y$$

10. (d) Given that l, G_1, G_2, G_3, n are in G.P

$$G_1 = lr, G_2 = l^2r, G_3 = l^3r, n = l^4r$$

$$\begin{aligned} \text{Then } G_1^4 + 2G_2^4 + G_3^4 &= (lr)^4 + 2(l^2r)^4 + (l^3r)^4 \\ &= (l^4)(l^4) + 2l^4(l^4)^2 + l^4(l^4)^3 \\ &= l^8 \cdot n + 2l^8 \cdot n^2 + l^8n^3 = l^8(n^2 + 2nl + n^2) = l^8(n+l)^2 = 4m^2nl \end{aligned}$$

11. (a) HM of a and b is $\frac{2ab}{a+b}$

$$\Rightarrow \frac{2ab}{a+b} = \frac{a^n + b^n}{a^{n-1} + b^{n-1}}$$

$$\Rightarrow 2a^n b + 2a b^n = a^{n+1} + b^{n+1} + a^n b + a b^n$$

$$\Rightarrow a^n b + a b^n = a^{n+1} + b^{n+1}$$

$$\Rightarrow a^n b - a^{n+1} = b^{n+1} - a b^n$$

$$\Rightarrow a^n(b-a) = b^n(b-a)$$

$$\Rightarrow \frac{a^n}{b^n} = 1$$

$$\Rightarrow n = 0$$

12. (b) Given: $a + ar + ar^2 = 14$

$$\Rightarrow a = \frac{14}{1+r+r^2} \quad \dots\dots(1)$$

$a + 1, ar + 1, \text{ and } ar^2 - 1$ are in arithmetic sequence as given

$$\Rightarrow ar + 1 - a - 1 = ar^2 - 1 - ar - 1$$

Simplifying and using (1) we get

$$2r^2 - 5r + 2 = 0$$

Solving this quadratic equation

$$r = 2 \text{ or } 1/2$$

The geometric sequence for $r = 2$ is 2, 4, 8

The geometric sequence for $r = 1/2$ is 8, 4, 2

So, the lowest term is 2.

13. (b) Let it happens after n months.

$$3 \times 200 + \frac{n-3}{2} [2 \times 240 + (n-4)40] = 11040$$

$$\Rightarrow \left(\frac{n-3}{2}\right) (480 + 40n - 160) = 11040 - 600 = 10440$$

$$\Rightarrow n^2 + 5n - 546 = 0 \Rightarrow (n+26)(n-21) = 0 \therefore n = 21$$

14. (a) In AP form 2nd term - 1st term

$$b^2 - a^2 = c^2 - b^2$$

$$b^2 + b^2 = c^2 + a^2$$

$$2b^2 = c^2 + a^2$$

Add $(2ab + 2ac + 2bc)$ on both sides

$$2b^2 + 2ab + 2ac + 2bc = a^2 + c^2 + ac + ac + bc + bc + ab + ab$$

$$2b^2 + 2ab + 2ac + 2bc = ac + bc + a^2 + ab + bc + c^2 + ab + ac$$

$$2b^2 + 2ab + 2ca + 2bc = ca + cb + a^2 + ab + cb + c^2 + ab + ac$$

$$2(ba + b^2 + ca + cb) = (ca + cb + a^2 + ab) + (cb + c^2 + ab + ac)$$

$$2((ba + b^2) + (ca + cb)) = ((ca + cb) + (a^2 + ab)) + ((cb + c) + (ab + ac))$$

$$2(b(a+b) + c(a+b)) = (c(a+b) + a(a+b)) + (c(b+c) + a(b+c))$$

$$2(b+c)(a+b) = (c+a)(a+b) + (c+a)(b+c)$$

Divide whole by $(a+b)(b+c)(c+a)$

$$\frac{2}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$$

$$\frac{1}{c+a} + \frac{1}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$$

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

2nd term - 1st term = 3rd term - 2nd term

Thus $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

15. (c) AM of a and b is $\frac{a+b}{2}$

$$\frac{a+b}{2} = 1 + i \cot\left(\frac{\pi}{2n}\right)$$

$$(a^n a + a^n b + b^n a + b^n b) = (2a^n a + 2b^n b)$$

$$(a^nb + b^na) = (a^na + b^nb)$$

$$a^n(b - a) = b^n(b - a)$$

$$a^n = b^n$$

This is possible only when $n = 0$

16. (d) Put $n = 2$ and $n = 1$, and we get

$$S_2 = 2P + Q$$

$$S_1 = P$$

$$\text{Therefore, } T_1 = S_1 = P$$

$$\text{Therefore, } T_2 = S_2 - S_1 = P + Q$$

$$\text{Therefore, the common difference } d = T_2 - T_1 = P + Q - P = Q$$

17. (c) Let the G.P. be a, ar, ar^2, ar^3, \dots

$$\text{we have } a + ar = 12 \quad \dots \text{(i)}$$

$$ar^2 + ar^3 = 48 \quad \dots \text{(ii)}$$

$$\text{On division we have } \frac{ar^2(1+r)}{a(1+r)} = \frac{48}{12} \Rightarrow r^2 = 4 \\ \therefore r = \pm 2$$

But the terms are alternately positive and negative, $\therefore r = -2$

$$\text{Now } a = \frac{12}{1+r} = \frac{12}{1-2} = \frac{12}{-1} = -12 \quad (\text{From (i)})$$

$$\begin{aligned} \text{18. (b)} \quad & 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \dots + 99^2 - 100^2 \\ & = (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \dots \end{aligned}$$

$$+ (99-100)(99+100)$$

$$= -(1+2) - (3+4) - (5+6) \dots - (99+100)$$

$$= -(1+2+3+4+5+6+\dots+99+100) = -\left(\frac{100}{2}(1+100)\right) \\ \left(\because S_n = \frac{n}{2}(\text{first term} + \text{last term})\right)$$

$$= -50 \times 101 = -5050.$$

19. (b) $F(1) = 2$

$$F(n+1) = \frac{2F(n)+1}{2}$$

$\Rightarrow F(2) = 2 \frac{1}{2}, F(3) = 3, F(4) = 3 \frac{1}{2} \dots$ which is in AP

Here, $a = 2, d = \frac{1}{2}$

$$\text{So, } F(101) = a_{101} = a + 100d = 2 + 100 \times \frac{1}{2} = 2 + 50 = 52$$

20. (b) For non trivial solution the determinant of the coefficient of various term vanish

$$\text{i.e. } \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$$\Rightarrow (3bc - 4bc) - 2a(c - b) + a(4c - 3b) = 0$$

$$\Rightarrow \frac{2ac}{a+c} = b$$

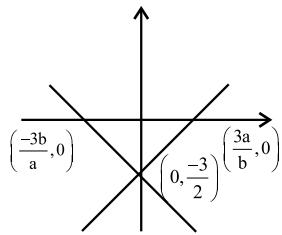
$$\Rightarrow a, b, c \in H.P.$$

11. Straight Lines

1. (b) Intercepts made by the lines with co-ordinate axis is

$$\left(\frac{-3b}{a}, 0\right), \left(0, \frac{-3}{2}\right) \text{ and } \left(0, \frac{-3}{2}\right)$$

Common intercepts is $\left(0, \frac{-3}{2}\right)$



2. (b) Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$... (1)

This passes through (3, 4), therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \quad \dots (2)$$

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting

$b = 14 - a$ in (2), we get

$$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow a^2 - 13a + 42 = 0$$

$$\Rightarrow (a-7)(a-6) = 0 \Rightarrow a = 7, 6$$

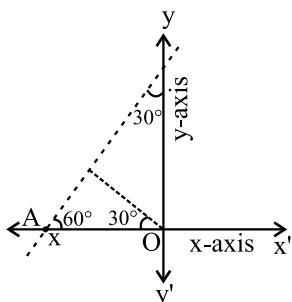
For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$.

Putting the values of a and b in (1), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1$$

or $x + y = 7$ and $4x + 3y = 24$

3. (a) Here $p = 7$ and $\alpha = 30^\circ$



∴ Equation of the required line is

$$x \cos 30^\circ + y \sin 30^\circ = 7$$

$$\text{or } x \frac{\sqrt{3}}{2} + y \times \frac{1}{2} = 7$$

$$\text{or } \sqrt{3}x + y = 14$$

4. (a) Let the equations of the line be $\frac{x}{a} + \frac{y}{b} = 1$, then the coordinates of point of intersection of this line and x-axis and

y-axis are respectively $(a, 0)$, $(0, b)$. Hence mid point of the intercept is $(a/2, b/2)$.

$$\begin{aligned} \therefore a/2 &= x_1 \Rightarrow a = 2x_1 \text{ and } b/2 = y_1 \\ &\Rightarrow b = 2y_1 \end{aligned}$$

Hence required equation of the line is

$$\begin{aligned} \frac{x}{2x_1} + \frac{y}{2y_1} &= 1 \\ \Rightarrow \frac{x}{x_1} + \frac{y}{y_1} &= 2 \end{aligned}$$

5. (b) The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line passing through $(2, 3)$ and making an angle of 45° is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$$\left[\text{Using } \frac{x-x_1}{\cos \theta} = \frac{y-y_1}{\sin \theta} = r \right]$$

co-ordinates of any point on this line are

$$(2 + r \cos 45^\circ, 3 + r \sin 45^\circ) \text{ or } \left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$$

If this point lies on the line $2x - 3y + 9 = 0$,

$$\text{then } 4 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0$$

$$\Rightarrow r = 4\sqrt{2}.$$

So the required distance = $4\sqrt{2}$.

6. (c) Substituting the coordinates of points A, B and C in the expression $x + 2y - 3$, we get-

The value of expression for A is

$$= -1 + 6 - 3 = 2 > 0$$

The value of expression for B is

$$= 2 - 6 - 3 = -7 < 0$$

The value of expression for C is

$$= 4 + 18 - 3 = 19 > 0$$

∴ Signs of expressions for A, C are same while for B, the sign of expression is different

∴ A, C are on one side and B is on other side of the line

7. (c) Third side passes through $(1, -10)$ so let its equation be $y + 10 = m(x - 1)$

If it makes equal angle, say θ with given two sides, then

$$\tan \theta = \frac{m - 7}{1 + 7m} = \frac{m - (-1)}{1 + m(-1)} \Rightarrow m = -3 \text{ or } 1/3$$

Hence possible equations of third side are

$$y + 10 = -3(x - 1) \text{ and } y + 10 = \frac{1}{3}(x - 1)$$

$$\text{or } 3x + y + 7 = 0 \text{ and } x - 3y - 31 = 0$$

8. (c) Slope of the given lines are $-1, -3, -\frac{1}{3}$ respectively

$$\text{Let } m_1 = -\frac{1}{3}, m_2 = -1, m_3 = -3$$

$$\therefore \tan A = \frac{-\frac{1}{3} + 1}{1 + \frac{1}{3} \cdot 1} = A = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\tan B = \frac{-1 + 3}{1 + 1 \cdot 3} \Rightarrow B = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\text{and } \tan C = \frac{-\frac{1}{3} + 1}{1 + 3 \cdot \frac{1}{3}} \Rightarrow C = \tan^{-1}\left(-\frac{4}{3}\right)$$

$\therefore \angle A = \angle B$, Hence triangle is isosceles triangle.

9. (b) Let m_1 and m_2 be the slopes of BA and BC respectively. Then

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2} \text{ and } m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let θ be the angle between BA and BC. Then

$$\begin{aligned} \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}} \right| = \left| \frac{\frac{10}{8}}{\frac{15}{8}} \right| = \pm \frac{2}{3} \\ \Rightarrow \theta &= \tan^{-1}\left(\frac{2}{3}\right) \end{aligned}$$

10. (b) Let the equation of sides AB, BC, CD and DA of parallelogram ABCD are respectively

$$y = \frac{3}{4}x + \frac{1}{4} \quad \dots(1); \quad y = \frac{3}{4}x + \frac{3}{4} \quad \dots(2)$$

$$y = \frac{4}{3}x - \frac{1}{3} \quad \dots(3); \quad y = \frac{4}{3}x - \frac{2}{3} \quad \dots(4)$$

$$\text{Here } m = \frac{3}{4}, n = \frac{4}{3}, a = \frac{1}{4}, b = \frac{3}{4}, c = -\frac{1}{3}, d = -\frac{2}{3}$$

\therefore Area of parallelogram ABCD

$$= \left| \frac{(a-b)(c-d)}{m-n} \right| = \left| \frac{\left(\frac{1}{4}-\frac{3}{4}\right)\left(-\frac{1}{3}+\frac{2}{3}\right)}{\frac{3}{4}-\frac{4}{3}} \right|$$

$$= \left| \frac{-\frac{1}{2} \times \frac{1}{3}}{-\frac{7}{12}} \right| = \frac{2}{7}$$

11. (c) Equation of a line parallel to $ax + by + c = 0$ is written as

$$ax + by + k = 0 \quad \dots(1)$$

If it passes through (c, d), then

$$ac + bd + k = 0 \quad \dots(2)$$

Subtracting (2) and (1), we get

$$a(x - c) + b(y - d) = 0$$

Which is the required equation of the line.

12. (b) Let the line L cut the axes at A and B say.

$$OA = a, OB = b$$

$$\therefore \text{Area } \Delta OAB = \frac{1}{2}ab = 5 \quad \dots(1)$$

Now equation of line perpendicular to lines

$$5x - y = 1 \text{ is } x + 5y = k$$

Putting $x = 0, y = b, y = 0, x = k = a$

$$\therefore \frac{1}{2}k \cdot k / 5 = 5 \quad \text{from....(1)}$$

$$k^2 = 50 \Rightarrow k = 5\sqrt{2}$$

$$\text{Hence the required line is } x + 5y = \pm 5\sqrt{2}$$

Note : Trace the line approximately and try to make use of given material as per the question.

13. (c) Solving for A,

$$x + 2y - 3 = 0$$

$$5x + y + 12 = 0$$

$$\Rightarrow \frac{x}{+24+3} = \frac{y}{-15-12} = \frac{1}{-9}$$

$$\therefore A(-3, 3)$$

Similarly B(1,1), C(1,-1), D(-2,-2)

Now m_1 = slope of AC = -1

m_2 = slope of BD = 1

$m_1 m_2 = -1 \quad \therefore$ the angle required is 90°

14. (b) If the lines are concurrent, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$

$$\Rightarrow (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow a+b+c = 0$$

$$[\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \neq 0]$$

$$15. (a) \text{Slope of BC} = \frac{-3+1}{-1+3} = -1$$

Now equation of line parallel to BC is

$$y = -x + k \Rightarrow y + x = k$$

Now length of perpendicular from O on this line

$$= \pm \frac{k}{\sqrt{2}} = \frac{1}{2} \Rightarrow k = -\frac{\sqrt{2}}{2}$$

\therefore Equation of required line is

$$2x + 2y + \sqrt{2} = 0$$

- 16. (a)** Let the required line by method $P + \lambda Q = 0$ be
 $(x - 3y + 1) + \lambda(2x + 5y - 9) = 0$
 \therefore perpendicular from $(0, 0) = \sqrt{5}$ gives

$$\frac{1-9\lambda}{\sqrt{(1-2\lambda)^2 + (5-3\lambda)^2}} = \sqrt{5},$$

squaring and simplifying $(8\lambda - 7)^2 = 0$

$$\Rightarrow \lambda = 7/8$$

Hence the line required is

$$(x - 3y + 1) + 7/8(2x + 5y - 9) = 0$$

$$\text{or } 22x + 11y - 55 = 0 \Rightarrow 2x + y - 5 = 0$$

Note: Here to find the point of intersection is not necessary.

- 17. (a)** Let equation of variable line is

$$ax + by + c = 0 \quad \dots(1)$$

Now sum of perpendicular distance

$$\frac{2a+c}{\sqrt{a^2+b^2}} + \frac{2b+c}{\sqrt{a^2+b^2}} + \frac{a+b+c}{\sqrt{a^2+b^2}} = 0$$

$$\Rightarrow a + b + c = 0 \quad \dots(2)$$

on subtracting (2) from (1), we get

$$a(x-1) + b(y-1) = 0$$

Which obviously passes through a fixed point P(1, 1).

- 18. (c)** Here equation of bisectors

$$\frac{3x-4y+7}{5} = \pm \frac{12x+5y-2}{13}$$

Which give, $11x - 3y + 9 = 0$ and

$$21x + 77y - 101 = 0$$

Now angle between the line $3x - 4y + 7 = 0$ and one bisector

$$11x - 3y + 9 = 0$$

$$|\tan \theta| = \left| \frac{-9+44}{33+12} \right| = \left| \frac{35}{45} \right| < 1$$

Hence the bisector is the required.

$$11x - 3y + 9 = 0$$

- 19. (c)** Given lines will be concurrent if

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0 \Rightarrow -bc + 2ac - ab = 0$$

$$\Rightarrow b = \frac{2ac}{a+c}$$

$\Rightarrow a, b, c$ are in H.P.

- 20. (a)** Here the sides $x + y - 5 = 0$ and $x - y + 1 = 0$ are perpendicular to each other, therefore $y = 1$ will be hypotenuse of the triangle. Now its middle point will be the circumcentre.

Now solving the pair of equations

$$x + y - 5 = 0, y - 1 = 0$$

$$\text{and } x - y + 1 = 0, y - 1 = 0, \text{ we get}$$

$$P \equiv (4, 1), Q \equiv (0, 1)$$

$$\text{Mid point of PQ or circumcentre} = (2, 1)$$

12. Conic Sections

1. (b) The hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ can be rewritten in the following way: $\frac{x^2}{5} - \frac{y^2}{5 \cos^2 \alpha} = 1$

This is the standard form of a hyperbola, where $a^2 = 5$ and $b^2 = 5 \cos^2 \alpha$.

$$\Rightarrow b^2 = a^2 (e_1^2 - 1)$$

$$\Rightarrow 5 \cos^2 \alpha = 5 (e_1^2 - 1)$$

$$\Rightarrow e_1^2 = \cos^2 \alpha + 1 \quad \dots(1)$$

- The ellipse $x^2 \sec^2 \alpha + y^2 = 25$ can be rewritten in the following way: $\frac{x^2}{25 \cos^2 \alpha} + \frac{y^2}{25} = 1$

This is the standard form of an ellipse, where $a^2 = 25$ and $b^2 = 25 \cos^2 \alpha$.

$$b^2 = a^2 (1 - e_2^2)$$

$$\Rightarrow e_2^2 = 1 - \cos^2 \alpha$$

$$\Rightarrow e_2^2 = \sin^2 \alpha \quad \dots(2)$$

According to the questions,

$$\cos^2 \alpha + 1 = 3 (\sin^2 \alpha)$$

$$\Rightarrow 2 = 4 \sin^2 \alpha$$

$$\Rightarrow \sin \alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \alpha = \frac{\pi}{4}$$

2. (a) Combined equation of the asymptotes is

$$(x + 2y - 3)(3x + 4y + 5) = 0$$

\therefore Equation of the hyperbola can be taken as

$$(x + 2y + 3)(3x + 4y + 5) + k = 0$$

Given the hyperbola is passing through p(1, -1)

$$\Rightarrow (1 - 2 + 3)(3 - 4 + 5) + k = 0$$

$$\Rightarrow 8 + k = 0 \Rightarrow k = -8$$

Equation of the hyperbola is

$$(x + 2y + 3)(3x + 4y + 5) - 8 = 0$$

$$3x^2 + 6xy + 9x + 4xy + 8y^2 + 12y + 5x + 10y + 15 - 8 = 0$$

$$3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$$

3. (a) Length of latus rectum of each parabola = 3

\therefore Equation of parabolas be $y^2 = 3x$... (i)

$$\text{and } x^2 = 3y \quad \dots(\text{ii})$$

$$\text{The equation of tangent to (i) is } y = mx + \frac{3}{4m} \quad \dots(\text{iii})$$

(iii) is also tangent to $x^2 = 3y$

$$\Rightarrow x^2 = 3mx + \frac{9}{4m} \quad 4mx^2 = 12m^2x + 9 \Rightarrow 4mx^2 - 12m^2x - 9 = 0$$

$$\text{Now } D = 0 \Rightarrow 144m^4 - 4(-9)(4m) = 0 \Rightarrow m(m^3 + 1) = 0$$

$$\Rightarrow m = 0 \text{ (which is not possible)} \therefore m = -1$$

Put $m = -1$ in (iii), we get $y = -x - \frac{3}{4}$

or $4(x + y) + 3 = 0$, which is the required equation of tangent.

4. (d) $y^2 = 8x$

$$\Rightarrow y^2 = 4(2x)$$

We know that equation of tangent of slope 'm' is

$$y = mx + \frac{a}{m}$$

$$\Rightarrow y = mx + \frac{2}{m}$$

This tangent also touches $xy = -1$

$$\text{Hence, put } y = mx + \frac{2}{m}$$

$$\Rightarrow x \left(mx + \frac{2}{m} \right) = -1$$

$$\Rightarrow mx^2 + \frac{2x}{m} = -1$$

$$\Rightarrow mx^2 + 2x + m = 0$$

As tangent touches curve at one point, hence discriminant is 0

$$\Rightarrow 2^2 - 4 \times m \times m = 0$$

$$\Rightarrow 4m^2 = 4$$

$$\Rightarrow m^2 = 1$$

$$y = mx + \frac{2}{m}$$

$$\Rightarrow y = x + 2$$

5. (c)

6. (a) Circles and are said to intersect orthogonally if $2g_1g_2 + 2f_1f_2 = C_1 + C_2$

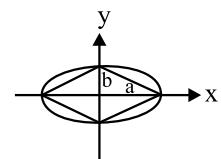
$$\Rightarrow 2a\alpha + 2b\beta = \beta^2 + \alpha^2$$

7. (d) We have, $e = \frac{3}{5}$

$$2ae = 6 \Rightarrow a = 5$$

As $b^2 = a^2(1 - e^2)$

$$\Rightarrow b^2 = 25 \left(1 - \frac{9}{25} \right) \Rightarrow b = 4$$



$$\therefore \text{Area of quadrilateral} = 4 \left(\frac{1}{2ab} \right)$$

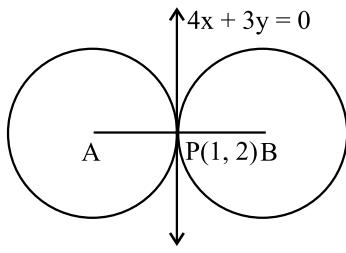
$$= 2ab = 2 \times 5 \times 4 = 40 \text{ sq. units}$$

8. (c)

9. (b) The equation of the common tangent is

$$4x + 3y - 10 = 0 \quad \dots(\text{i})$$

As the two circles are each of radius 5 units, the circles touch externally at the point P(1, 2). The centres A, B of the required circles lie on a line perpendicular to (i) and pass through P(1, 2).



Slope of line (i) = $-\frac{4}{3}$, so slope of line AB = $\frac{3}{4}$.

If θ is the inclination of the line AB, then

$$\tan \theta = \frac{3}{4}$$

$$\Rightarrow \sin \theta = \frac{3}{5} \text{ and } \cos \theta = \frac{4}{5}$$

($\because 0 \leq \theta < \pi$ and $\tan \theta$ is +ve, so θ lies in first quadrant)

The equation of the line AB in distance form is

$$\frac{x-1}{\cos \theta} = \frac{y-2}{\sin \theta} = r \quad \text{i.e. } \frac{x-1}{\frac{4}{5}} = \frac{y-2}{\frac{3}{5}} = r \quad \dots\dots(\text{ii})$$

Since $AP = 5 = BP$, putting $r = 5, -5$ in (ii), the coordinates of the points B and A are $\left(1 + \frac{4}{5} \cdot 5, 2 + \frac{3}{5} \cdot 5\right)$ and $\left(1 + \frac{4}{5}(-5), 2 + \frac{3}{5}(-5)\right)$ i.e. $(5, 5)$ and $(-3, -1)$

The equation of the required circles are

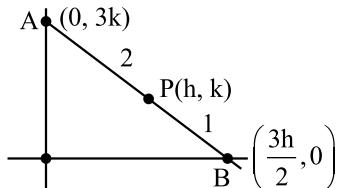
$$(x-5)^2 + (y-5)^2 = 5^2 \text{ and } (x+3)^2 + (y+1)^2 = 5^2$$

$$\text{i.e. } x^2 + y^2 - 10x - 10y + 25 = 0 \text{ and } x^2 + y^2 + 6x + 2y - 15 = 0.$$

- 10. (a)** A line $y = mx + c$ is normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2 m^2}$.

Eccentric angle is $\frac{\pi}{3}$ so $3(a^2 m^2 + b^2) = 4c^2$

- 11. (c)** Let AB be the given ladder as shown in the figure below and let P(h, k) be the given point on ladder such that PB = 4 units and PA = 8 units. Thus P divides AB in the ratio 2 : 1.



\therefore Co-ordinates of A will be $(0, 3k)$ and that of B will be $\left(\frac{3h}{2}, 0\right)$

$$\text{Now, } AB = 12 \Rightarrow \frac{9h^2}{4} + 9k^2 = 144$$

$$\text{So, locus of P is } \frac{x^2}{64} + \frac{y^2}{16} = 1$$

$$12. (\text{c}) \text{ We have, } \frac{x^2}{12} + \frac{y^2}{16} = 1 \quad \therefore e = \sqrt{1 - \frac{12}{16}} = \frac{1}{2}$$

Foci $\equiv (0, 2)$ & $(0, -2)$

So, transverse axis of hyperbola = $2b = 4 \Rightarrow b = 2$

$$\& a^2 = b^2(e^2 - 1) \Rightarrow a^2 = 4\left(\frac{9}{4} - 1\right) \Rightarrow a^2 = 5$$

$$\therefore \text{ Required equation is } \frac{x^2}{5} - \frac{y^2}{4} = -1$$

- 13. (b)** Equation of line joining points ' α ' and ' β ' is

$$\frac{x}{a} \cos \frac{\alpha + \beta}{2} + \frac{y}{b} \sin \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

If it is a focal chord, then it passes through focus $(ae, 0)$, so

$$e \cos \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

$$\Rightarrow \frac{\cos \frac{\alpha - \beta}{2}}{\cos \frac{\alpha + \beta}{2}} = \frac{e}{1}$$

$$\Rightarrow \frac{\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta}{2}} = \frac{e-1}{e+1}$$

$$\Rightarrow \frac{2 \sin \alpha / 2 \sin \beta / 2}{2 \cos \alpha / 2 \cos \beta / 2} = \frac{e-1}{e+1}$$

$$\Rightarrow \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{e-1}{e+1}$$

- 14. (a)** It is given that coordinates of Foci are S(-1, 0) and S'(7, 0).

therefore SS' = 2ae

$$2ae = 7 - (-1) = 8$$

$$ae = 4$$

$$a \times \frac{1}{2} = 4 \quad [\text{if is given that eccentricity } e = \frac{1}{2}]$$

therefore $a = 8$

$$\text{since } b^2 = a^2(1 - e^2)$$

$$b^2 = 8^2 \left(1 - \frac{1}{2^2}\right)$$

$$= 64 \times \left(1 - \frac{1}{4}\right)$$

$$= 64 \times \frac{3}{4}$$

$$b^2 = 48$$

$$\Rightarrow b = 4\sqrt{3}$$

centre of the ellipse is given by $\left(\frac{-1+7}{2}, \frac{0+0}{2}\right)$ i.e. $(3, 0)$

therefore equation of the ellipse is

$$\frac{(x-3)^2}{8^2} + \frac{y^2}{(4\sqrt{3})^2} = 1$$

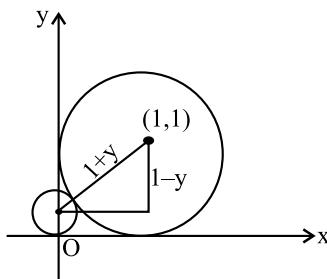
therefore parametric representation is

$$x - 3 = a \cos \theta = 8 \cos \theta \Rightarrow x = 3 + 8 \cos \theta$$

$$y - 0 = b \sin \theta = 4\sqrt{3} \sin \theta \Rightarrow y = 4\sqrt{3} \sin \theta$$

15. (c) We have by Pythagoras theorem $(1+y)^2 = (1-y)^2 + 1$

$$\Rightarrow 4y = 1 \quad \therefore y = \frac{1}{4}$$



16. (c) $(2y-5)^2 = -2x - 17 + 25 = -2x + 8 = -2(x-4)$

$$\text{or}, \left(y - \frac{5}{2}\right)^2 = \frac{-1}{2} \times (x-4)$$

So Latus rectum is $\frac{1}{2}$.

17. (c)

18. (a)

19. (a) Let (h, k) be the co-ordinate of centroid

$$\therefore h = \frac{a \cos t + b \sin t + 1}{3}, k = \frac{a \sin t - b \cos t + 0}{3}$$

$$\Rightarrow 3h - 1 = a \cos t + b \sin t \quad \dots\dots(i)$$

$$3k = a \sin t - b \cos t \quad \dots\dots(ii)$$

Squaring (i) and (ii) then adding, we get

$$(3h-1)^2 + (3k)^2 = a^2(\cos^2 t + \sin^2 t) + b^2(\cos^2 t + \sin^2 t)$$

Replacing (h,k) by (x,y) we get choice (a) is correct.

20. (a) Let centre be $(-h, -k)$ equation

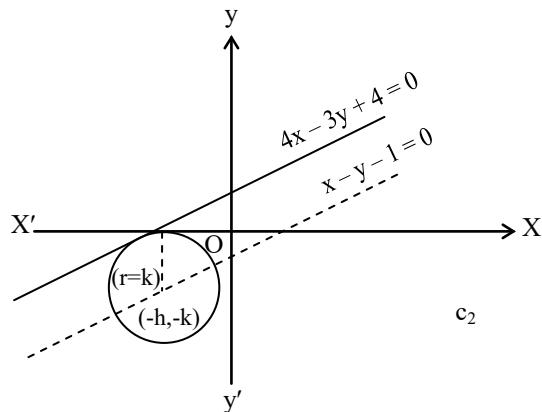
$$(x+h)^2 + (y+k)^2 = k^2 \quad \dots\dots(1)$$

$$\text{Also } -h+k=1 \quad \dots\dots(2)$$

$\therefore h=k-1$ radius $= k$ (touches x-axis)

Touches the line $4x-3y+4=0$

$$\left| \frac{-4h-3(-k)+4}{5} \right| = k \quad \dots\dots(3)$$



$$\text{Solving (2) and (3), } h = \frac{1}{3}, k = \frac{4}{3}$$

Hence the circle is

$$\left(x - \frac{4}{5}\right)^2 + \left(y + \frac{4}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

$$\Rightarrow 9(x^2 + y^2) + 6x + 24y + 1 = 0$$

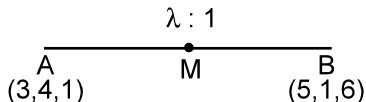
13. Introduction to 3D Geometry

1. (a) D divides BC in the ratio AB : AC i.e. 3 : 13. Therefore coordinates of D are

$$\left(\frac{3 \times -9 + 13 \times 5}{3+13}, \frac{3 \times 6 + 13 \times 3}{3+13}, \frac{3 \times -3 + 13 \times 2}{3+13} \right)$$

$$\text{or } \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16} \right)$$

2. (c) Let the line joining A and B crosses the xy-plane at the point M and let M divides AB in the ratio $\lambda : 1$ internally.



Co-ordinates of M are

$$\left[\frac{5\lambda + 3}{\lambda + 1}, \frac{\lambda + 4}{\lambda + 1}, \frac{6\lambda + 1}{\lambda + 1} \right]$$

Since the point M lies on the xy-plane

Its z-co-ordinate is zero.

$$\therefore \frac{6\lambda + 1}{\lambda + 1} = 0 \Rightarrow 6\lambda + 1 = 0$$

$$\text{or } \lambda = -\frac{1}{6}$$

The ratio is $\lambda : 1 = \frac{1}{6} : 1$ i.e. 1 : 6 externally.

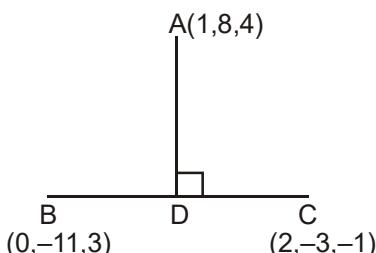
Hence the co-ordinates of M are

$$\text{i.e. } \left[\frac{5(-1/6)+3}{(-1/6)+1}, \frac{(-1/6)+4}{(-1/6)+1}, 0 \right],$$

$$\left(\frac{13}{5}, \frac{23}{5}, 0 \right)$$

3. (b) Let D be the foot of the \perp from A on BC, If P divides BC in the ratio $k : 1$, then co-ordinates of D are

$$\left(\frac{2k+0}{k+1}, \frac{-3k-11}{k+1}, \frac{-k+3}{k+1} \right)$$



D.R.'s of AD are

$$\frac{2k}{k+1}-1, \frac{-3k-11}{k+1}-8, \frac{-k+3}{k+1}-4$$

$$\Rightarrow \frac{k-1}{k+1}, \frac{-11k-19}{k+1}, \frac{-5k-1}{k+1}$$

D.R.'s of BC are $2-0, -3+11, -1-3$,

i.e., $2, 8, -4$

Since $AD \perp BC$

$$\therefore \frac{k-1}{k+1} \times 2 + \frac{-11k-19}{k+1} \times 8 + \frac{-5k-1}{k+1} \times (-4) = 0$$

$$\Rightarrow \frac{2k-2-88k-152+20k+4}{k+1} = 0$$

$$\Rightarrow -66k - 150 = 0 \Rightarrow 66k = -150$$

$$\therefore k = -\frac{25}{11}$$

Hence the co-ordinates of D are $\left(\frac{25}{7}, \frac{23}{7}, -\frac{4}{7} \right)$

4. (c) The centroid of the triangle is -

$$\left(\frac{2+3-2}{3}, \frac{-4-1+5}{3}, \frac{3-2+8}{3} \right) \equiv (1, 0, 3)$$

Its distance from x-axis = $\sqrt{0^2 + 3^2} = 3$

5. (a)

6. (b)

7. (b) Since direction cosines of line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c} \right)$

$$\therefore \frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1 \Rightarrow c^2 = 3 \Rightarrow c = \pm \sqrt{3}$$

8. (b) $AB = \sqrt{4+9+36} = 7$

$$BC = 7$$

$$CD = 7$$

$$DA = 7$$

$$AC = \sqrt{16+25+81}$$

$$= \sqrt{122}$$

$$BD = \sqrt{64+1+9}$$

$$= \sqrt{74}$$

rhombus.

9. (d) $\overrightarrow{AB} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$$\overrightarrow{BC} = -4\hat{i} - 6\hat{j} - 8\hat{k}$$

$$\overrightarrow{BC} = -2\overrightarrow{AB}$$

Collinear point.

10. (a) $\frac{2+\lambda(3)}{\lambda+1} = 0$

$$\lambda = -\frac{2}{3}$$

11. (b) $\lambda : 1$

$$2\left(\frac{1+4\lambda}{1+\lambda}\right) - 3\left(\frac{2-5\lambda}{\lambda+1}\right) + \left(\frac{-1+2\lambda}{\lambda+1}\right) = 4$$

$$\Rightarrow 2 + 8\lambda - 6 + 15\lambda - 1 + 2\lambda = 4\lambda + 4$$

$$21\lambda = 9$$

14. Limits and Derivatives

1. (b) $\lim_{x \rightarrow 1} (1 - x + [x - 1] + [1 - x])$

$$\text{L.H.L.} = \lim_{x \rightarrow 1^-} (1 - x - 1) = -1$$

$$\text{R.H.L.} = \lim_{x \rightarrow 1^+} (1 - x - 1) = -1$$

$$\text{L.H.L.} = \text{R.H.L.} = -1$$

2. (a) $\lim_{x \rightarrow \infty} \frac{x^4 + x^2 + 1}{x^5 + x^2 - 1}$

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2} + \frac{1}{x^4}}{x + \frac{1}{x^2} - \frac{1}{x^4}} = 0$$

3. (b) $\lim_{x \rightarrow 0} (1 + x)^{1/13x}$

$$e^{\lim_{x \rightarrow 0} \frac{1}{13x}(1+x-1)} = e^{1/13}$$

4. (b) $fa. = ga. = k$

$$\lim_{x \rightarrow a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4$$

differentiating n times

$$\lim_{x \rightarrow a} \frac{f(a)g^n(x) - g(a)f^n(x)}{g^n(x) - f^n(x)} = 4$$

since $ga. = fa. = k$

$$\lim_{x \rightarrow a} \frac{k(g^n(x) - f^n(x))}{g^n(x) - f^n(x)} = 4$$

$$\Rightarrow k = 4$$

5. (d) $\lim_{n \rightarrow \infty} \frac{[x] + [2x] + [3x] + \dots + [nx]}{1 + 2 + 3 + \dots + n}$

$$x - 1 \leq [x] \leq x$$

$$2x - 1 \leq [2x] \leq 2x$$

⋮

$$nx - 1 \leq [nx] \leq nx$$

$$(x - 1) + (2x - 1) + \dots + (nx - 1) \leq [x] + [2x]$$

$$+ \dots + [nx] \leq x + 2x + \dots + nx$$

$$\frac{(x + 2x + 3x + \dots + nx) - n}{1 + 2 + \dots + n}$$

$$\lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{L_1} \leq \lim_{n \rightarrow \infty}$$

$$\frac{[x] + [2x] + \dots + [nx]}{1 + 2 + \dots + n} \leq \lim_{n \rightarrow \infty} \frac{x + 2x + \dots + nx}{1 + 2 + \dots + n}$$

$$L_1 = \lim_{n \rightarrow \infty} \frac{(x + 2x + \dots + nx) - n}{1 + 2 + \dots + n}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{(x)(n)(n+1)}{2} - n}{\frac{n(n+1)}{2}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{xn(n+1) - 2n}{n(n+1)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x \left(1 + \frac{1}{n}\right) - \frac{2}{n}}{\left(1 + \frac{1}{n}\right)} = x$$

$$L_2 = \lim_{n \rightarrow \infty} \frac{x + 2x + 3x + \dots + nx}{1 + 2 + 3 + \dots + n}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{(x)(n)(n+1)}{2}}{\frac{(n)(n+1)}{2}} = x$$

$$\Rightarrow L_1 = L_2 = L = x$$

6. (b) $S = \frac{3}{1^3} + \frac{5}{1^3 + 2^3} + \frac{7}{1^3 + 2^3 + 3^3} + \dots$

$$t_r = \frac{2r+1}{1^3 + 2^3 + \dots + r^3}$$

$$= \frac{2r+1}{r^2(r+1)^2}$$

$$S = \lim_{n \rightarrow \infty} \sum_{r=1}^n t_r$$

$$t_r = \frac{8r+4}{r^2(r+1)^2}$$

$$= 4 \left[\frac{1}{r^2} - \frac{1}{(r+1)^2} \right]$$

$$S = \lim_{n \rightarrow \infty} 4 \left[\frac{1}{1} - \frac{1}{4} \right]$$

$$+ \frac{1}{4} - \frac{1}{9}$$

$$+ \frac{1}{9} - \frac{1}{16}$$

⋮

$$+ \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \lim_{n \rightarrow \infty} 4 \left[1 - \frac{1}{(n+1)^2} \right] = 4$$

7. (b) $\lim_{x \rightarrow -1^+} \frac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}$

Let $x = \cos \theta$
as $x \rightarrow -1^+$, $\theta \rightarrow \pi^-$

$$\lim_{x \rightarrow \pi^-} \frac{\sqrt{\pi} - \sqrt{\theta}}{\sqrt{2 \cos \theta / 2}} \left(\begin{array}{l} 0 \\ 0 \end{array} \right)$$

using L Hospital rule

$$\lim_{x \rightarrow \pi^-} \frac{-\frac{1}{2\sqrt{\theta}}}{-\frac{\sqrt{2} \sin \theta}{2}} = \frac{1}{\sqrt{2\pi}}$$

8. (a) $\lim_{n \rightarrow \infty} \left(an - \frac{1+n^2}{1+n} \right) = b$

$$\lim_{n \rightarrow \infty} \left(\frac{an + an^2 - 1 - n^2}{1+n} \right) = b$$

$$\lim_{n \rightarrow \infty} \frac{a - 1/n + n(a-1)}{1 + 1/n} = b$$

Ordered pair must be (1,1)

∴ Option (a) is correct Answer

9. (a) $\lim_{x \rightarrow 0} \frac{2x - \left(x + \frac{x^3}{6} + \dots \right)}{2x - \left(x - \frac{x^3}{3} + \dots \right)} = \frac{1}{3}$

10. (d) $\lim_{x \rightarrow 1^-} \{x\} = \lim_{x \rightarrow 1^-} (x - [x]) = 1 - 0 = 1$

$$\lim_{x \rightarrow 1^-} \{x\} = (x - [x]) = 1 - 1 = 0$$

$$\therefore \lim_{x \rightarrow 1^-} \frac{x \sin \{x\}}{x-1} = \lim_{x \rightarrow 1^-} \frac{x}{x-1} \sin \{x\} = -\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x \sin \{x\}}{\{x\}} \frac{x-1}{x-1} = 1 \times 1 \times 1 = 1$$

Since, L.H. limit ≠ R. H. limit

11. (b) $\tan \theta = \cot \theta - 2 \cot 2\theta$

$$\therefore \frac{1}{2} \tan \frac{\theta}{2} = \frac{1}{2} \cot \frac{\theta}{2} - \cot \theta$$

$$\frac{1}{2^n} \tan \frac{\theta}{2^n} = \frac{1}{2^n} \cot \frac{\theta}{2^n} - \frac{1}{2^{n-1}} \cot \frac{\theta}{2^{n-1}}$$

$$\therefore \text{Required limit} = \lim_{n \rightarrow \infty}$$

$$S_n = \lim_{n \rightarrow \infty} \left(\frac{1}{\frac{2^n \tan \theta/2^n}{\theta/2^n} \cdot \frac{\theta}{2^n}} - 2 \cot 2\theta \right) = \frac{1}{\theta} - 2 \cot 2\theta$$

12. (b) Writing the given expression in the form

$$\left(\frac{\sin x^n}{x^n} \right) \left(\frac{x^n}{x^m} \right) \left(\frac{x}{\sin x} \right)^m \text{ and noting that the } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1,$$

we see that the required limit equals to 1, if $n = m$, and

0 if $n > m$.

13. (c) $2^x + 2^y = 2^{x+y}$

$$\frac{dy}{dx} = -\frac{(2^x \ln 2 - 2^y 2^x \ln 2)}{(2^y \ln 2 - 2^y 2^x \ln 2)}$$

$$= -2^{x-y} \left[\frac{1-2^y}{1-2^x} \right]$$

$$\frac{dy}{dx} = 2^{x-y} \left[\frac{2^y - 1}{1-2^x} \right]$$

14. (c) $y = x |x|$

$$y' = 2|x|$$

15. (c)

$$\frac{dy}{dx} = \frac{a \cos \theta}{a \sec^2 \frac{\theta}{2} - a \sin \theta + \frac{\tan \frac{\theta}{2}}{2}}$$

$$\frac{a \cos \theta}{-a \sin \theta + \frac{a}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}}}$$

$$= \frac{a \cos \theta}{\frac{a}{\sin \theta} - a \sin \theta}$$

$$= \frac{a \cos \theta \sin \theta}{2a - a \sin^2 \theta}$$

$$= \frac{a \cos \theta \sin \theta}{a \cos^2 \theta}$$

$$= \tan \theta$$

16. (a) $y = x^{\log x}$

$$y' = \log x (x)^{\log x - 1} + \frac{x^{\log x} \log x}{x}$$

$$= \frac{2 \log x \ x^{\log x}}{x}$$

17. (a) $\frac{(\log x)}{(\log_{1/5} x)}$

$$\log_{1/5} x = \frac{\log x}{\log 1/5}$$

$$\frac{d}{dx} (\log_{1/5} x) = \frac{1}{\log 1/5} \times \frac{1}{x}$$

$$\frac{y}{x} = \log 1/5$$

18. (a) $ax^2 + 2hxy + by^2 = 0$

$$\frac{dy}{dx} = -\left(\frac{2ax + 2hy}{2hx + 2by} \right)$$

$$= -\frac{(ax + hy)}{(hx + by)}$$

$$\begin{aligned}
 19. \text{ (b)} \quad & y = x \log \left(\frac{x}{a+bx} \right) = x(\ln x - \ln(a+bx)) \\
 \Rightarrow & y' = \ln \left(\frac{x}{a+bx} \right) + x \left[\frac{1}{x} - \frac{b}{a+bx} \right] \\
 \Rightarrow & y' = \frac{y}{x} + 1 - \frac{bx}{a+bx} \\
 \Rightarrow & y' = \frac{y}{x} + \frac{a}{a+bx} \quad \dots\dots(1) \\
 y'' = & \frac{y'x-y}{x^2} - \frac{ab}{(a+bx)^2} \\
 = & \frac{y'x-y}{x^2} - \frac{x(bx+a-a)}{x(a+bx)^2} \\
 = & \frac{y'x-y}{x^2} - \frac{a}{x} \left[\frac{1}{(a+bx)} - \frac{a}{(a+bx)^2} \right] \\
 = & \frac{y'x-y}{x^2} - \frac{a}{x(a+bx)} + \frac{a^2}{x(a+bx)^2}
 \end{aligned}$$

use (1)

$$\begin{aligned}
 y'' &= \frac{y'x-y}{x^2} - \frac{1}{x} \left\{ y' - \frac{y}{x} - \left(y' - \frac{y}{x} \right)^2 \right\} \\
 &= \frac{y'}{x} - \frac{y}{x^3} + \frac{y}{x} + \frac{y}{x^2} + \left(\frac{y'x-y}{x^3} \right)^2 \\
 x^3y'' &= (xy' - y)^2
 \end{aligned}$$

$$20. \text{ (b)} \quad x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \left(\frac{t^2 + 1}{t^2 - 1} \right) \\
 \frac{d^2y}{dx^2} &= \frac{(t^2 - 1)2t - (t^2 + 1)2t}{(t^2 - 1)^2 \left(1 - \frac{1}{t^2} \right)} \\
 \frac{d^2y}{dx^2} &= \frac{-4t^3}{(t^2 - 1)^3}
 \end{aligned}$$

15. Mathematical Reasoning

- 1. (c)
- 2. (d)
- 3. (c)
- 4. (b)
- 5. (a)
- 6. (a)
- 7. (c)
- 8. (c)
- 9. (d)
- 10. (c)
- 11. (b, d)
- 12. (a)
- 13. (d)
- 14. (c)
- 15. (d)
- 16. (c)
- 17. (d)
- 18. (c)
- 19. (b)

20. (c) Given $(p \wedge \sim q) \wedge (p \wedge r) \rightarrow \sim p \vee q$ is false

$\Rightarrow (p \wedge \sim q \wedge r) \rightarrow \sim p \vee q$ is false

$\Rightarrow (\sim p \wedge q \wedge \sim r) \vee \sim p \vee q$ is false

$\Rightarrow (\sim p \vee q \vee \sim r) \vee (\sim p \vee q)$ is false

$\Rightarrow \sim p \vee p \vee \sim r$ is false

So, truth values of $\sim p$, q and $\sim r$ must be F, F, F.

Thus, truth values of p , q and r must be T, F, T.

16. Statistics

1. (b) Total student = 100

$$\text{For 70 students total marks} = 75 \times 70 \\ = 5250$$

$$\text{Total marks of girls} = 7200 - 5250 \\ = 1950$$

$$\text{Average marks of girls} = \frac{1950}{30} \\ = 65$$

2. (c) Sum of 100 items = $49 \times 100 = 4900$

$$\text{Sum of wrong items} = 38 + 22 + 50 = 110$$

$$\text{Sum of correct items} = 60 + 70 + 80 = 210$$

$$\therefore \text{correct sum} = 4900 - 110 + 210 = 5000$$

$$\therefore \text{correct mean} = \frac{5000}{100} = 50$$

3. (c) Let the unknown weight be x .

$$\text{Mean} = \frac{52 + 58 + 55 + 53 + 56 + 54 + x}{7} = 55$$

$$\Rightarrow 328 + x = 385$$

$$\Rightarrow x = 57$$

4. (d) The standard deviation is independent of change of origin, So, put $x_i - 5 = y_i$

$$\therefore \text{Given equation become } \sum_{i=1}^9 y_i = 9 \text{ and } \sum_{i=1}^9 y_i^2 = 45$$

$$SD = \sqrt{\frac{1}{n} \sum y_i^2 \left(\frac{\sum y_i}{n} \right)^2} = \sqrt{\frac{45}{9} - \left(\frac{9}{9} \right)^2} = \sqrt{5-1} = \sqrt{4} = 2$$

5. (d) Given that, $\sum x_i^2 = 400$ and $\sum x_i = 80$. If $x_i > 0$.

$i = 1, 2, 3, \dots, n$, which are not identical, then using the inequality.

AM of m^{th} power > m^{th} power of AM, if $m > 1$

$$\Rightarrow \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} > \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^2$$

[Here, $m = 2$]

$$\Rightarrow \frac{\sum_{i=1}^n x_i^2}{n} > \left(\frac{\sum_{i=1}^n x_i}{n} \right)^2$$

$$\Rightarrow \frac{400}{n} > \left(\frac{80}{n} \right)^2 \Rightarrow \frac{n^2}{n} > \frac{80^2}{400}$$

$$\Rightarrow n > 16$$

$$6. (c) \text{ Median} = \frac{Q_1 + Q_2}{2} = \frac{25 + 45}{2} = 35$$

7. (a) The descending order is

$$\alpha + 5, \alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha - 2, \alpha - \frac{5}{2}, \alpha - 3, \alpha - \frac{7}{2}, \alpha - 4$$

$N = 8$ is even

$$\text{So, median} = \frac{\alpha - 2 + \alpha - \frac{5}{2}}{2} = \alpha - \frac{9}{4}$$

8. (c) Let the other two observations be x and y .

According to question, Mean = 5

$$a + b + 9 = 25 \Rightarrow a + b = 16 \quad \dots(i)$$

Also, variance = 124

$$\frac{\sum x_i^2}{n} - (\bar{x})^2 = 124 \Rightarrow \frac{1+4+36+a^2+b^2}{5} - 25 = 124$$

$$\Rightarrow a^2 + b^2 = 704 \quad \dots(ii)$$

$$\Rightarrow (a+b)^2 - 2ab = 704 \Rightarrow -2ab = 448$$

$$\Rightarrow \text{Now, } (a-b)^2 = a^2 + b^2 - 2ab = 704 + 448$$

$$\Rightarrow (a-b)^2 = 1152 \Rightarrow a - b = 24\sqrt{2} \quad \dots(iii)$$

On solving (i) and (iii), we get

$$a = 8 + 12\sqrt{2}, \quad b = 8 - 12\sqrt{2}$$

$$\therefore \text{Mean deviation about mean} = \frac{\sum |x_i - \bar{x}|}{n}$$

$$= \frac{|1-5| + |2-5| + |8+12\sqrt{2}-5| + |18-12\sqrt{2}-5|}{5} = \frac{14}{5} = 2.8$$

9. (c) $n = 31$ so by definition

10. (d) Since $n = 9$, the median term = $\frac{9+1}{2} = 5^{\text{th}}$ term.

Last four observations ($6^{\text{th}}, 7^{\text{th}}, 8^{\text{th}}$ and 9^{th}) are increased by 2. The median is 5^{th} observation which remains unchanged.

$$11. (a) \text{ We have } \alpha^2 = \frac{\sum x_i^2}{h} - \left(\frac{\sum x_i}{h} \right)^2$$

$$\sum x_i^2 = 2^2 + 4^2 + \dots + 100^2 = 4\{1^2 + 2^2 + \dots + 50^2\} = \frac{4 \times 50 \times 51 \times 101}{6}$$

$$\therefore \frac{\sum x_i^2}{50} = \frac{4 \times 51 \times 101}{6} \quad 3434 \left(\frac{\sum x_i}{50} \right)^2 = \left(\frac{2 \times 50 \times 51}{2 \times 50} \right)^2 = 2601$$

$$\Rightarrow \alpha^2 = 3434 - 2601 = 833$$

$$12. (b) \text{ Mode} = 3 \text{ Median} - 2 \text{ mean} = 3 \times 22 - 2 \times 21 = 66 - 42 = 24$$

$$13. (d) \text{ Coefficient of variation} = \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{19.76}{35.16} \times 100.$$

14. (a) Let x and y are number of boys and girls in a class respectively.

$$\frac{52x + 42y}{x+y} = 50 \Rightarrow x = 4y \Rightarrow \frac{x}{y} = \frac{4}{1} \text{ and } \frac{x}{x+y} = \frac{4}{5}$$

$$\text{Required percentage} = \frac{x}{x+y} \times 100 = \frac{4}{5} \times 100 = 80\%$$

$$15. (d) \text{ Mean} = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

$$SD = \sqrt{\frac{1+2^2+3^2+4^2+5^2+6^2}{6} - \left(\frac{7}{2}\right)^2} = \sqrt{\frac{35}{12}}$$

$$\Rightarrow x = 4, 7$$

$$\Rightarrow y = 7, 4$$

16. (c) Let the other two observations be x and y.

$$\text{Mean} = 4$$

$$\Rightarrow x + y = 11$$

$$\text{Variance} = 5.2$$

$$\Rightarrow \frac{1}{5} \sum (x_i - 4)^2 = 5.2$$

$$\Rightarrow \frac{1}{5} [(1-4)^2 + (2-4)^2 + (6-4)^2 + (x-4)^2 + (y-4)^2] = 5.2$$

$$\Rightarrow x^2 + y^2 = 65$$

$$(x+y)^2 = x^2 + y^2 + 2xy$$

$$\Rightarrow 121 = 65 + 2xy$$

$$\Rightarrow y = \frac{28}{x}$$

$$x + y = 11$$

$$\Rightarrow x + \frac{28}{x} = 11$$

$$\Rightarrow x^2 - 11x + 28 = 0$$

So, the numbers are 4 and 7

17. (c) It is obvious.

18. (a) Given: $\Sigma x = 170$, $\Sigma x^2 = 2830$ and increase in $\Sigma x = 10$

$$\Rightarrow \Sigma x' = 170 + 10 = 180$$

$$\text{Increase in } \Sigma x^2 = 900 - 400 = 500$$

$$\Rightarrow \Sigma x'^2 = 2830 + 500 = 3330$$

$$\text{Correct Variance} = \frac{1}{n} \Sigma x'^2 - \left(\frac{1}{n} \Sigma x' \right)^2$$

$$= \frac{1}{15} \times 3330 - \left(\frac{1}{15} \times 180 \right)^2$$

$$= 222 - 144$$

$$= 78$$

19. (a) The ascending order is 7, 10, 12, 15, 17, 19, 25

$$QD = \frac{Q_3 - Q_1}{2} = \frac{10 + 19}{2} = 14.5$$

20. (c) Mode = 3 median - 2 mean

$$= 3 \times 22 - 2 \times 21 = 3(22 - 14) = 3 \times 8 = 24$$

17. Probability

1. (d) Using the wavy curve method, given inequality is satisfied for $x < 20$ or $30 < x < 40$.

\therefore Number of favourable outcomes = 28

$$\text{Required probability} = \frac{28}{100} = \frac{7}{25}$$

2. (c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore P(A \cap B) = \frac{1}{3} + \frac{3}{4} - \frac{5}{6} = \frac{3}{12} = \frac{1}{4}$$

$$P(A) P(B) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

As $P(A) P(B) = P(A \cup B)$ \therefore independent

$P(A \cap B) \neq 0$ \therefore not exclusive.

3. (c) $P(\text{non occurrence of } (A_i)) = 1 - (i+1) = \frac{i}{i+1}$.

$\therefore P(\text{non occurrence of any of events})$

$$= \left(\frac{1}{2} \right) \cdot \left(\frac{2}{3} \right) \cdots \left\{ \frac{n}{(n+1)} \right\} = \frac{1}{(n+1)}$$

4. (a) Let the number of marble be $2n$ (where n is a large number)

$$\begin{aligned} \text{Required probability} &= \lim_{n \rightarrow \infty} \frac{n \times {}^n C_4}{2^n C_5} \times \frac{{}^n C_3 \times {}^n C_2}{2^n C_5} \\ &= \lim_{n \rightarrow \infty} \frac{n \times n(n-1)(n-2)(n-3)}{4!} \times \frac{n(n-1)(n-2)}{3!} \\ &\quad \times \frac{n(n-1)}{2!} \times \frac{(5)^2 [(2n-5)!]^2}{(2n!)^2} \\ &\lim_{n \rightarrow \infty} \frac{n^4 (n-1)^3 (n-2)^2 (n-3) [(2n-5)!]^2 \times 5 \times 5 \times 4 \times 3!}{3! 2! (2n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{50n^4 (n-1)^3 (n-2)^2 (n-3)}{[2n(2n-1)(2n-2)(2n-3)(2n-4)]^2} = \frac{50}{1024} = \frac{25}{512} \end{aligned}$$

5. (a) Any element of A has four possibilities: element belongs to (i) both P_1 and P_2 (ii) Neither P_1 nor P_2 (iii) P_1 but not P_2 (iv) P_2 but not P_1 . Thus $n(S) = 4^n$. For the favourable cases, we choose one element in n ways and this element has three choices as (i), (iii) and (iv), while the remaining $n-1$ elements have one choice each, namely (ii).

$$\text{Hence required probability} = \frac{3n}{4^n}$$

6. (a) Let \bar{A} be the event of having different birthdays. Each can have birthday in 365 ways, so N persons can have their birthdays in 365^N ways. Number of ways in which all have different birthdays = ${}^{365}P_N$

$$\therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{{}^{365}P_N}{(365)^N} = 1 - \frac{(365)!}{(365)^N (365-N)!}$$

7. (a) $P(E \cap F) = P(E) P(F) = \frac{1}{12}$ (i)

$$P(E^c \cap F^c) = P(E^c) P(F^c) = \frac{1}{2}$$

$$\Rightarrow [1 - P(E)][1 - P(F)] = \frac{1}{2} \quad \dots(ii)$$

Solving (i) and (ii), we get $P(E) = \frac{1}{3}$ & $P(F) = \frac{1}{4}$ as $P(E) > P(F)$.

8. (b) Let S be the sample space and let E be the required event, then $n(S) = ({}^{52}C_2)^2$. For the number of elements in E , we first choose a card (which we want common) and then from the remaining cards (51 in numbers) we choose two cards and distribute them among A and B in $2!$ ways. Hence $n(E) = {}^{52}C_1 \cdot {}^{51}C_2 \cdot 2!$. Thus $P(E) = \frac{50}{663}$.

9. (b) 19 numbers (x_1, \dots, x_{19}) are less than x_{20} .

30 numbers (x_{21}, \dots, x_{50}) are more than x_{20} .

x_{20} will be in middle if two numbers are randomly picked from each of the set (x_1, \dots, x_{19}) and (x_{21}, \dots, x_{50}) .

Number of ways of selecting five numbers that have x_{20} in middle = ${}^{19}C_2 \times {}^{30}C_2$

Total number of ways of selecting any five numbers = ${}^{50}C_5$

$$\text{Required probability} = \frac{{}^{19}C_2 \times {}^{30}C_2}{{}^{50}C_5}$$

10. (b) Required probability

$$= \left(\frac{3}{4} \right) \left(\frac{1}{2} \right) \left(\frac{3}{8} \right) + \left(\frac{1}{4} \right) \left(\frac{1}{2} \right) \left(\frac{3}{8} \right) + \left(\frac{3}{4} \right) \left(\frac{1}{2} \right) \left(\frac{3}{8} \right) = \frac{21}{64}$$

11. (c) Required probability = $1 - [P(\text{No Head}) + P(\text{No Tail})]$

$$= 1 - \left\{ \frac{1}{2^8} + \frac{1}{2^8} \right\} = 1 - \frac{1}{2^7} = 1 - \frac{1}{128} = \frac{127}{128}$$

12. (b) Total cases when $a + b > c$ are

$\{(1,1,1), (2,2,1), (2,2,2), (2,2,3), (3,3,1), \dots, (3,3,5), (4,4,1), \dots, (4,4,6), (5,5,6), (6,6,1), \dots, (6,6,6)\} = 27$

$$\therefore \text{Required probability} = \frac{1}{27}$$

13. (b) Possibility of getting 9 are $(5, 4), (4, 5), (6, 3), (3, 6)$

$$\text{Probability of getting score 9 in a single throw} = \frac{4}{36} = \frac{1}{9}$$

$$\text{Probability of getting score 9 exactly twice} = {}^3C_2 \left(\frac{1}{9} \right)^2 \times \left(\frac{8}{9} \right) = \frac{8}{243}$$

14. (b) Total case ${}^nC_r = \frac{n!}{r!(n-r)!} = {}^{20}C_2 = \frac{20 \cdot 19}{2} = 190$

Prime no.s = 2, 3, 5, 7, 11, 13, 17, 19 = 8

Favourable case = ${}^8C_2 = 28$

$$\text{Required probability} = \frac{28}{190} = \frac{14}{95}$$

15. (c) PROBABILITY

vowel = O, A, I, I

$$\text{Total case} = {}^{11}C_1 = 11$$

$$\text{Favourable case} = {}^4C_1 = 4$$

$$\text{Required probability} = \frac{4}{11}$$

$$16. \text{ (b)} \text{ Total case} = {}^8C_3 = 56$$

$$\text{Favourable case } x = {}^4C_3 = 4$$

$$\text{unfavourable case } y = 56 - 4 = 52$$

$$\text{odds against} = \frac{y}{x} = \frac{52}{4} = \frac{13}{1} = 13 : 1$$

$$17. \text{ (a)} \text{ odds against } E_1 = \frac{7}{4}$$

$$\text{probability of } E_1 = \frac{4}{11}$$

$$\text{odds against } E_2 = \frac{5}{3}$$

$$\text{probability of } E_2 = \frac{3}{8}$$

$$\text{Probability of } E_3 = 1 - [P(E_1) + P(E_2)]$$

$$= 1 - \left[\frac{4}{11} + \frac{3}{8} \right] = 1 - \frac{65}{88} = \frac{23}{88}$$

$$\text{odds against } E_3 = \frac{88 - 23}{23} = \frac{65}{23} = 65 : 23$$

$$18. \text{ (b)} \text{ Total case} = 52$$

$$\text{Favourable case } x = 13 + 3 = 16$$

$$\text{unfavourable case } y = 52 - 16 = 36$$

$$\text{odds in favour} = \frac{x}{y} = \frac{16}{36} = \frac{4}{9} = 4 : 9$$

$$19. \text{ (a)} \frac{P(A)}{1-P(A)} = \frac{1}{3} \Rightarrow P(A) = \frac{1}{4}$$

$$\frac{P(B)}{1-P(B)} = \frac{1}{4} \Rightarrow P(B) = \frac{1}{5}$$

$$\frac{P(C)}{1-P(C)} = \frac{1}{5} \Rightarrow P(C) = \frac{1}{6}$$

$$\frac{P(D)}{1-P(D)} = \frac{1}{6} \Rightarrow P(D) = \frac{1}{7}$$

$$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$$

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}$$

$$20. \text{ (b)} P(A) = \frac{75}{100} = \frac{3}{4}, \quad P(B) = \frac{60}{100} = \frac{3}{5}$$

$$\text{Required probability} = 1 - P(\bar{A}) P(\bar{B})$$

$$= 1 - \left(\frac{1}{4} \cdot \frac{2}{5} \right) = 1 - \frac{1}{10} = \frac{9}{10}$$

CLASS XII

1. Relations and Functions

1. (d) We know that for a relation to be function every element of first set should be associated with one and only one element of second set but elements of first set can have same f-image in second set which is given in d.

$$2. \text{ (d)} \quad f(y) = \frac{y}{y-1} = \frac{(x-1)/x}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x} = 1-x.$$

$$3. \text{ (a)} \quad \text{Domain} = \{x; x \in \mathbb{R}; x^3 - x \neq 0\} \\ = \mathbb{R} - \{-1, 0, 1\}$$

4. (c) [x] is an integer, $\cos(-x) = \cos x$ and

$$\cos\left(\frac{\pi}{2}\right) = 0, \cos 2\left(\frac{\pi}{2}\right) = -1$$

$$\cos 0\left(\frac{\pi}{2}\right) = 1, \cos 3\left(\frac{\pi}{2}\right) = 0, \dots$$

Hence range = $\{-1, 0, 1\}$

$$5. \text{ (d)} \quad (f+g)(x) = f(x) + g(x) \\ = x^2 + 2 + \sqrt{x+1}$$

$$6. \text{ (a)} \quad f(x) = \frac{1}{x^2} + \frac{1}{x^3} = \frac{x+1}{x^3}$$

= ratio of two polynomials

$\therefore f(x)$ is a rational function.

$$7. \text{ (b)} \quad \text{Here } |\sin 2x| = \sqrt{\sin^2 2x} \\ = \sqrt{\frac{(1-\cos 4x)}{2}}$$

Period of $\cos 4x$ is $\pi/2$

Period of $|\sin 2x|$ will be $\pi/2$.

$$8. \text{ (a)} \quad \text{Here } f\{f(x)\} = f\left(\frac{x-3}{x+1}\right) = \frac{\left(\frac{x-3}{x+1}\right)-3}{\left(\frac{x-3}{x+1}\right)+1} = \frac{x+3}{1-x}$$

$$\therefore f[f\{f(x)\}] = \frac{\frac{x+3}{1-x}-3}{\frac{x+3}{1-x}+1} = \frac{4x}{4} = x$$

$$9. \text{ (c)} \quad 2 < x < 3 \Rightarrow |x-2| = x-2$$

$$|x-3| = 3-x$$

$$f(x) = 2(x-2) - 3(3-x) = 5x - 13.$$

$$10. \text{ (c)} \quad x_1 \neq x_2 \Rightarrow e^{x_1} \neq e^{x_2}$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

$\therefore f(x) = e^x$ is one-one.

$$11. \text{ (d)} \quad \because 4 \neq -4, \text{ but } f(4) = f(-4) = 16$$

$\therefore f$ is many one function.

Again $f(\mathbb{R}) = \mathbb{R}^+ \cup \{0\}$ \mathbb{R} , therefore f is into.

12. (b) Observing the graph of this function, we find that every line parallel to x-axis meets its graph at more than one point so it is not one-one.

Now range of $f = \mathbb{N}$ = Co-domain, so it is onto.

13. (a) $A = \{2, 4, 6\} : B = \{2, 3, 5\}$

$\therefore A \times B$ contains $3 \times 3 = 9$ elements.

Hence, number of relations from A to B = 2^9 .

14. (a) Since $8^n - 7n - 1 = (7+1)^n - 7n - 1$

$$= 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \dots + {}^nC_{n-1} 7 + {}^nC_n - 7n - 1 \\ = {}^nC_2 7^2 + {}^nC_3 7^3 + \dots + {}^nC_n 7^n, ({}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1} \text{ etc.}) \\ = 49[{}^nC_2 + {}^nC_3(7) + \dots + {}^nC_n 7^{n-2}]$$

$\therefore 8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$

$$\text{For } n = 1, 8^n - 7n - 1 = 8 - 7 - 1 = 0$$

$$\text{For } n = 2, 8^n - 7n - 1 = 64 - 14 - 1 = 49$$

$\therefore 8^n - 7n - 1$ is a multiple of 49 for all $n \in \mathbb{N}$.

$\therefore X$ contains elements which are multiples of 49 and clearly Y contains all multiples of 49. $\therefore X \subseteq Y$.

15. (b) $N_3 \cap N_4 = \{3, 6, 9, 12, 15, \dots\} \cap \{4, 8, 12, 16, 20, \dots\}$

$$= \{12, 24, 36, \dots\} = N_{12}.$$

Trick : $N_3 \cap N_4 = N_{12}$

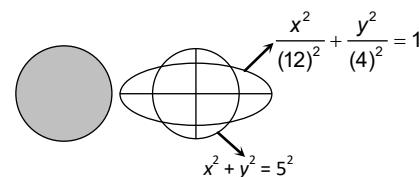
[$\because 3, 4$ are relatively prime numbers].

16. (b) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - n(A \cap B)$

Since, maximum number of elements in $A \cap B = 3$

\therefore Minimum number of elements in $A \cup B = 9 - 3 = 6$.

17. (d) $A = \text{Set of all values } (x, y) : x^2 + y^2 = 25 = 5^2$



$$B = \frac{x^2}{144} + \frac{y^2}{16} = 1 \text{ i.e., } \frac{x^2}{(12)^2} + \frac{y^2}{(4)^2} = 1$$

Clearly, $A \cap B$ consists of four points.

18. (b) $A - B = A \cap B^c = A \cap \bar{B}$.

19. (a) Clearly, $A = \{2, 3\}, B = \{2, 4\}, C = \{4, 5\}$

20. (a) Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1)\}, S = \{(2, 2), (2, 3)\}$ be transitive relations on A.

Then $R \cup S = \{(1, 1); (1, 2); (2, 2); (2, 3)\}$

Obviously, $R \cup S$ is not transitive. Since $(1, 2) \in R \cup S$ and $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

2. Inverse Trigonometry Functions

1. (d) As we know, $\sin^{-1}(A) + \cos^{-1}(A) = \frac{\pi}{2}$

$$\Rightarrow x^2 + ax + \frac{\pi}{2} = 0$$

For atleast one solution $b^2 - 4ac \geq 0$

$$\Rightarrow a^2 - 4 \times 1 \times \frac{\pi}{2} \geq 0$$

$$\Rightarrow a^2 - 2\pi \geq 0 \Rightarrow a^2 \geq 2\pi$$

$$\Rightarrow a = \frac{-\pi}{1} \text{ or } a = -2 \frac{-\pi}{4}$$

2. (c) $\sin \frac{7\pi}{6} = \sin \left(\pi + \frac{\pi}{6} \right) = -\sin \left(\frac{\pi}{6} \right) = -\cos \left(\frac{\pi}{2} - \frac{\pi}{6} \right) = -\cos \left(\frac{\pi}{3} \right)$

$$\Rightarrow \cos^{-1} \left(-\sin \frac{7\pi}{6} \right) = \cos^{-1} \left(\cos \left(\frac{\pi}{3} \right) \right) = \frac{\pi}{3}$$

3. (b)

4. (a) $[\tan(\sin^{-1} x)]^2 > 1$

$$\Rightarrow \tan(\sin^{-1} x) < 1 \text{ or } \tan(\sin^{-1} x) > 1$$

$$\Rightarrow \frac{\pi}{4} < \sin^{-1} x < \frac{\pi}{2} \text{ or } -\frac{\pi}{2} < \sin^{-1} x < -\frac{\pi}{4}$$

$$\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1 \right) \cup x \in \left(-1, -\frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow x \in \left(-1, -\frac{1}{\sqrt{2}} \right) \cup \left(\frac{1}{\sqrt{2}}, 1 \right)$$

5. (c) We have, $2\tan^{-1}x + \sin^{-1} \frac{2x}{1+x^2}$ ($\because 2\tan^{-1}x = \sin^{-1} \frac{2x}{1+x^2}$)

$$= 2\tan^{-1}x + 2\tan^{-1}x = 4\tan^{-1}x$$

It is independent of x. So, $x \in (-\infty, -1] \cup [1, \infty)$

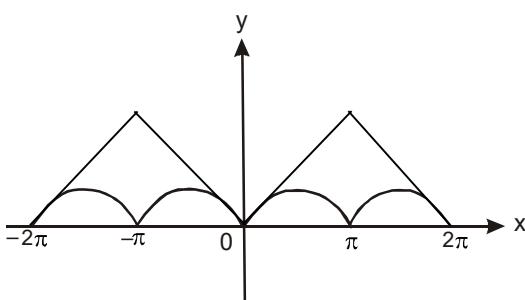
6. (b) $\ln [0, \pi], y = \cos^{-1}(\cos x) = x$

$$\text{In } [\pi, 2\pi], y = \cos^{-1} \{ \cos(2\pi - x) \} = 2\pi - x$$

$$\text{In } [-\pi, 0], y = \cos^{-1} \cos(-x) = -x$$

$$\text{In } [-2\pi, -\pi], y = \cos^{-1} \{ \cos(2\pi + x) \} = 2\pi + x$$

Plotting the graph, we have



required ordered pairs are

$(0, 0), (2\pi, 0)$ and $(-\pi, 0)$.

Hence required number is 3.

7. (b) $\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1} [-1]$

$$= \pi - \tan^{-1} [1]$$

$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$$

$$2(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = 2\pi$$

8. (a) $\cos^{-1} \frac{3}{5} = \theta \quad \therefore \frac{3}{5} = \cos \theta$

$$\therefore \frac{4}{5} = \sin \theta$$

$$\Rightarrow \sin \left[\cos^{-1} \left(\frac{3}{5} \right) \right] = \sin \theta = \frac{4}{5}$$

9. (c) because $\frac{7\pi}{6}$ does not lies in $(0, \pi)$

$$\text{so } \left(\cos \left(2\pi - \frac{5\pi}{6} \right) \right) \cos^{-1} = \cos^{-1} \left(\cos \left(\frac{5\pi}{6} \right) \right) = \frac{5\pi}{6}$$

10. (b) Let $\sin^{-1} x = y$

$$x = \sin y$$

$$2y^2 - y - 6 = 0$$

$$2y^2 - 4y + 3y - 6 = 0$$

$$(2y + 3)(y - 2) = 0$$

$$y = -1.5, 2$$

$$2 > \frac{\pi}{2}$$

So that only solution is -1.5 .

11. (a) We have

$$\sin^{-1} \left[\cot \left(\sin^{-1} \sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1} \frac{\sqrt{12}}{4} + \sec^{-1} \sqrt{2} \right) \right]$$

$$= \sin^{-1} \left[\cot \left(\sin^{-1} \sqrt{\frac{\sqrt{3}-1}{2\sqrt{2}}} + \cos^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} \frac{1}{\sqrt{2}} \right) \right]$$

$$= \sin^{-1} [\cot(15^\circ + 30^\circ + 45^\circ)]$$

$$= \sin^{-1} [\cot 90^\circ] = \sin^{-1} 0 = 0$$

12. (a) We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

$$(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x \left(\frac{\pi}{2} - \tan^{-1}x \right) = \frac{5\pi^2}{8}$$

$$\frac{\pi^2}{4} - 2 \frac{\pi}{2} \tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1}x)^2 - \tan^{-1}x \frac{3\pi^2}{8} = 0$$

$$\tan^{-1}x = -\frac{\pi}{4} \quad \Rightarrow x = -1$$

13. (c) We have, $\sin^{-1}x > \cos^{-1}x$

$$\Rightarrow \sin^{-1}x > \frac{\pi}{2} - \sin^{-1}x \quad 2\sin^{-1}x > \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x > \frac{\pi}{4}. \quad \Rightarrow \sin(\sin^{-1}x) > \sin \frac{\pi}{4} \quad x > \frac{1}{\sqrt{2}}$$

$$\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1 \right] \quad \text{since } -1 \leq x \leq 1$$

14. (c) $2 \sin^{-1} x = \sin^{-1} \left(2x \sqrt{1-x^2} \right)$

Range of right hand side is $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

$$\Rightarrow -\frac{\pi}{2} \leq 2 \sin^{-1} x \leq \frac{\pi}{2} \quad \Rightarrow -\frac{\pi}{4} \leq \sin^{-1} x \leq \frac{\pi}{4}$$

$$\Rightarrow x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

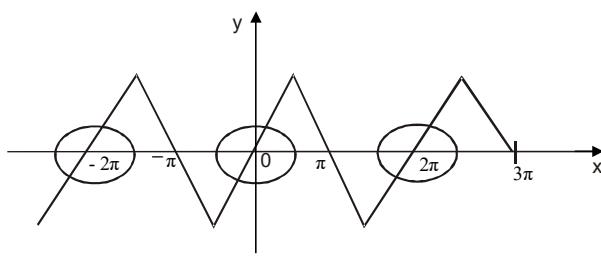
15. (b) $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$

$$\Rightarrow \sum \left(\frac{\pi}{2} - \tan^{-1} x \right) = \frac{\pi}{2} \Rightarrow \operatorname{Stan}^{-1} x = p$$

$$\Rightarrow (\operatorname{Stan}^{-1} x) = 0 \Rightarrow \operatorname{Stan}(\tan^{-1} x) = \prod \tan(\tan^{-1} x)$$

$$\Rightarrow x + y + z = xyz$$

16. (d) Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly five times in $[-2\pi, 3\pi]$



17. (c) $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{4}$$

18. (a) Let $\cos^{-1}\left(\frac{1}{8}\right) = \theta$, where $0 < \theta < \pi$, then $\frac{1}{2} \cos^{-1}$

$$\frac{1}{8} = \frac{1}{2} \theta$$

$$\Rightarrow \cos\left(\frac{1}{2} \cos^{-1} \frac{1}{8}\right) = \cos \frac{\theta}{2} \text{ now } \cos^{-1}\left(\frac{1}{8}\right) = \theta$$

$$\Rightarrow \cos \theta = \frac{1}{8} \Rightarrow \cos^2 \theta = \frac{9}{16} \Rightarrow \cos^2 \frac{\theta}{2} = \frac{3}{4}$$

19. (a) $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1} 3x - \tan^{-1} x$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2} \Rightarrow 4x^3 - x = 0 \Rightarrow x = 0, \pm \frac{1}{2}$$

None of which satisfies $1 < x < \sqrt{2}$

20. (c) Since $\frac{x}{y} \cdot \frac{x+y}{x-y} > 1$, then the equation is equal to

$$\begin{aligned} & \pi + \tan^{-1} \left[\frac{\frac{x}{y} + \frac{x+y}{x-y}}{1 - \frac{x}{y} \times \frac{x+y}{x-y}} \right] \\ &= \pi + \tan^{-1}(-1) = \frac{3\pi}{4} \end{aligned}$$

3. Matrices

1. (c) Trace of matrix is defined as

$$\sum_{i=1}^n a_{ii} = 2x^2 + 2x - 12 = 0 \Rightarrow x = -3, 2$$

2. (c) If $AB = 0$, then at least one of A and B is necessarily singular.

3. (c) It is a known fact that $(AB)' = B'A'$.

4. (b) Given $A^2 - B^2 = (A + B)(A - B)$

$$\Rightarrow 0 = BA - AB \Rightarrow BA = AB$$

$$5. (d) \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix} = 10A^{-1} \Rightarrow \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix} (A) = 10I$$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \dots\dots(1)$$

$$\Rightarrow -5 + \alpha = 0$$

Equating A_{21} entry on both sides of (Eq.1)

$$\Rightarrow \alpha = 5$$

$$6. (a) \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\sin \theta \cos \theta + \cos \theta \sin \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{So } \lambda = 1$$

$$7. (a) A^2 = AA = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

8. (c) A and B are matrices so $AB = BA$

$$A^2 = A \times A = A \times (BA) = A \times (AB) = B$$

$$B^2 = B \times B = B \times (AB) = B \times (BA) = A$$

$$\therefore A^2 + B^2 = A + B$$

$$9. (b) \text{ Here } aI + bA = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$\therefore (aI + bA)^2 = \begin{pmatrix} a^2 + 0 & ab + ba \\ 0 + 0 & 0 + a^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} = a^2 I + 2abA$$

10. (b) Here

$$AB = \begin{pmatrix} 1-6+2 & 4-3-4 & 1-3+2 & -3+4 \\ 2+2-3 & 8+1+6 & 2+1-3 & 1-6 \\ 4-6-1 & 16-3+2 & 4-1-3 & -3-2 \end{pmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

Also AC

$$= \begin{bmatrix} 2-9+4 & 1+6-10 & -1+3-2 & -2+3 \\ 4+3-6 & 2-2+15 & -2-1+3 & -4-1 \\ 8-9-2 & 4+6+5 & -4+3+1 & -8+3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix} = AB;$$

Hence $AC = AB$ is true

$$11. (a) \text{ Here } AB = \begin{bmatrix} pr - qs & ps + qr \\ -qr - ps & -qs + pr \end{bmatrix}$$

$$\text{Also } BA = \begin{bmatrix} rp - qs & qr + sp \\ -sp - qr & -qs + pr \end{bmatrix}$$

Clearly $AB = BA$

$$12. (c) \text{ Here } A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore A^2 - 4A = \begin{bmatrix} 9-4 & 8-8 & 8-8 \\ 8-8 & 9-4 & 8-8 \\ 8-8 & 8-8 & 9-4 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5I$$

$$13. (c) \text{ We have } (AB)_{11} = 1.3 + 2.1 = 5$$

$$(BA)_{11} = 3.1 + 4.3 = 15$$

$$\therefore AB \neq BA \text{ Again } (A^2)_{11} = 1.1 + 2.3$$

$$= 7 \neq 3 = (B)_{11}$$

$$\text{Also } (AB)^T = B^T A^T = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2 & 9+0 \\ 4+12 & 12+0 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix} \text{ is correct.}$$

$$14. (d) \text{ Here } AA^T = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(BB^T)_{11} = (4)^2 + (1)^2 \neq 1$$

$$(AB)_{11} = 8 - 7 = 1, (BA)_{11} = 8 - 7 = 1$$

$\therefore AB \neq BA$ may not be true

Now

$$AB = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

15. (c) $|A| = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = (4 \times 3 - 1 \times 2)$
 $= 12 - 2 = 10$

(\because if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$)

16. (c) Here $[A_{ij}] = \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 2 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} \end{bmatrix}$
 $= \begin{bmatrix} -24 & -27 & 30 \\ 4 & 1 & -2 \\ 8 & 11 & -10 \end{bmatrix}$ Hence transposing $[A_{ij}]$ we get

$$\text{adj } A = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$$

17. (a) $A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\therefore \alpha^2 = 1 \quad \& \quad \alpha + 1 = 5$$

\therefore Not possible.

18. (c) $|A| = \alpha^2 - 4$

$$\text{Now } |A|^3 = |A|^3 = 125$$

$$\therefore |A| = 5$$

$$\Rightarrow \alpha^2 - 4 = 5$$

$$\Rightarrow \alpha^2 = 9$$

$$\Rightarrow \alpha = \pm 3$$

19. (c) Given $6A^{-1} = A^2 + cA + dI$

$$\Rightarrow \frac{6}{6} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$+ \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix} + \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

$$\therefore 6 = 1 + c + d \quad \Rightarrow c + d = 5$$

$$\& -1 = 5 + c \quad \Rightarrow c = -6$$

$$\therefore d = 11$$

20. (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\& P^T P = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now $P^T \cdot Q^{2005} \cdot P = P^T \cdot Q \cdot Q \cdots Q \cdot P$

$$\underbrace{P^T \cdot}_{1} \underbrace{P \cdot}_{1} \underbrace{A \cdot}_{1} \underbrace{P^T \cdot}_{1} \underbrace{P \cdot}_{1} \underbrace{A \cdot}_{1} \underbrace{P^T \cdot}_{1} \underbrace{\cdots \cdots \cdots}_{1} \underbrace{P \cdot}_{1} \underbrace{A \cdot}_{1} \underbrace{P^T \cdot}_{1} \underbrace{P}_{1}$$

$$= IA \cdot IA \cdot IA \cdots IAI = A^{2005}$$

$$\therefore A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Similarly $A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

4. Determinants

1. (a) $A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 5\alpha + 5\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 25\alpha + 5\alpha^2 \\ 0 & 0 & 25 \end{bmatrix}$$

Given $|A^2| = 25, 625\alpha^2 = 25 \Rightarrow |\alpha| = \frac{1}{5}$

2. (c) KA is the matrix , in which all the entries of A are multiplied by K .

Hence $|KA| = K^n |A|$, taking K common from all the columns.

3. (d) $A' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{1 + \tan^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A' A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix} \Rightarrow |A' A^{-1}| = 1$$

4. (b) Applying $C_1 \rightarrow C_1 + C_2$ we get,

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4 \sin 2x \\ 2 & 1 + \cos^2 x & 4 \sin 2x \\ 1 & \cos^2 x & 1 + 4 \sin 2x \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ we get

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4 \sin 2x \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 2 + 4 \sin 2x \leq 6.$$

Hence maximum value is 6.

5. (c) $A^2 - A + I = O \Rightarrow I = A - A \cdot A$

$$IA^{-1} = AA^{-1} - A(AA^{-1}), A^{-1} = I - A.$$

6. (b) Given determinant is equal to ,

$$\begin{aligned} \tan A (\tan B \cdot \tan C - 1) - 1 (\tan C - 1) + 1 (1 - \tan B) \\ = \tan A \cdot \tan B \cdot \tan C - \tan A - \tan B - \tan C + 2 = 2 \end{aligned} \quad (\text{as } \Pi \tan A = \Sigma \tan A)$$

7. (d) Multiplication of R_1 by x , R_2 by y and R_3 by z , reduces the given determinant to ,

$$\frac{1}{xyz} \begin{vmatrix} 1 & x^3 & xyz \\ 1 & y^3 & xyz \\ 1 & z^3 & xyz \end{vmatrix} = \begin{vmatrix} 1 & x^3 & 1 \\ 1 & y^3 & 1 \\ 1 & z^3 & 1 \end{vmatrix} = 0$$

8. (c) $\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$

$$f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2 \cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

$$\text{Thus } \lim_{x \rightarrow 0} \frac{f(x)}{x} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

9. (d) $R_1 \rightarrow R_1 - R_2, R_2 \rightarrow -R_3$ reduces the determinant to,

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$$\begin{aligned} &\Rightarrow (p-a)(q-b)r + ((r-c)b) + (q-b)((r-c)a) = 0 \\ &\Rightarrow (p-a)((q-b)(r-c) - (r-c)a) = 0 \\ &\Rightarrow (p-a)((q-b)(r-c+c) - c(q-b-q)) - (q-b)(r-c) \\ &(p-a-p) = 0 \\ &\Rightarrow (p-a)(q-b)(r-c) + c(p-a)(q-b) - (p-a)q-b \\ &(r-c) + q(p-a)(r-c) - (p-a)(q-b)(r-c) + p(q-b) \\ &(r-c) = 0 \end{aligned}$$

Dividing through out by $(p-a)(q-b)(r-c)$ we get,

$$\begin{aligned} \frac{c}{r-c} + \frac{q}{q-b} + \frac{p}{p-a} &= 1 \\ \Rightarrow 1 + \frac{c}{r-c} + \frac{q}{q-b} + \frac{p}{p-a} &= 2 \\ \Rightarrow \frac{r}{r-c} + \frac{q}{q-b} + \frac{p}{p-a} &= 2 \end{aligned}$$

10. (d) As ω is root of unity, $\omega^3 = \omega^{3n} = 1$

$$\therefore \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^2 & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} = (\omega^{3n} - 1) - \omega^n (w^{2n} - \omega^{2n}) + \omega^{2n} (\omega^n - \omega^n) = 0$$

11. (b) $\Delta = \begin{vmatrix} x & 3 & 1 \\ 7 & 6 & z \\ 1 & y & 2 \end{vmatrix} \quad R_2 \rightarrow 100R_1 + 10R_3 + R_2$

$$\begin{vmatrix} x & 3 & 1 \\ 100x + 10 + 7 & 300 + 10y + 1 & 100 + 20 + z \\ 1 & y & 2 \end{vmatrix}$$

R_2 is multiple of k so Δ is multiple of k

12. (d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 4 \\ 2 & 1 & 7 \end{bmatrix}$

$$\Rightarrow \det A = 0$$

So infinitely many solutions.

13. (c) Applying $C_1 + C_2 + C_3$, we get

$$\text{Det.} = 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

[by $R_2 - R_1, R_3 - R_1$]

$$= 2(a+b+c)^3$$

14. (b) By the operation $C_3 - (\alpha C_1 + C_2)$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & ba + c & -(a\alpha^2 + 2ba + c) \end{vmatrix} = 0$$

$$\Rightarrow -(a\alpha^2 + 2ba + c)(ac - b^2) = 0$$

$$\Rightarrow b^2 = ac \Rightarrow a, b, c \text{ are in G.P.}$$

15. (c) Applying $R_1 - (R_2 + R_3)$, we get

$$\text{Det.} = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$$

(by $R_2 + R_1, R_3 + R_1$)

$$= 2(a^2b^2c^2 + a^2b^2c^2) = 4a^2b^2c^2$$

$$16. (a) \begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix} = \begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ c & z & 5 \end{vmatrix}$$

(changing rows into columns)

$$= \frac{1}{3} \begin{vmatrix} 3a & x & p \\ 3b & 2y & 5 \\ 3c & 3z & 15 \end{vmatrix}$$

$$= \frac{3}{3} \times \frac{1}{5} \begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = \frac{1}{5}(125) = 25$$

17. (b) Here the cofactors of $\lambda, c, -b, \dots$ in Δ are $a^2 + \lambda^2, ab + c\lambda, ca - b\lambda, \dots$ respectively.

Therefore the value of Δ' is Δ^2 .

18. (b) Breaking the given determinant into two determinants, we get

$$\begin{vmatrix} 3^2 + k & 4^2 & 3^2 + k \\ 4^2 + k & 5^2 & 4^2 + k \\ 5^2 + k & 6^2 & 5^2 + k \end{vmatrix} + \begin{vmatrix} 3^2 + k & 4^2 & 3 \\ 4^2 + k & 5^2 & 4 \\ 5^2 + k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 0 + \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0$$

[Applying $R_3 - R_2$ and $R_2 - R_1$ in second det.]

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 0 \quad [\text{Applying } R_3 - R_2]$$

$$\Rightarrow \begin{vmatrix} 9+k & 7-k & 3 \\ 7 & 2 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 0 \quad [\text{Applying } C_2 - C_1]$$

$$\Rightarrow 2(7-k-6) = 0$$

$$\Rightarrow k = 1.$$

19. (c) Expanding the det., we get

$$\begin{aligned} \Delta &= -(b-a)^2 [0 - (a-c)^2(c-b)^2] + (c-a)^2 \\ &\quad [(a-b)^2(b-c)^2 - 0] \\ &= 2(a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

20. (c) Applying $R_2 - R_1$ and $R_3 - R_1$, We get

$$\begin{vmatrix} 1 + \sin^2 \theta & \cos^2 \theta & 4 \sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2 + 4 \sin 4\theta = 0$$

$$\Rightarrow \sin 4\theta = -1/2$$

$$\Rightarrow 4\theta = n\pi + (-1)n(-\pi/6)$$

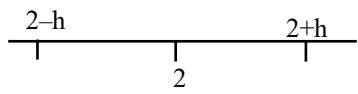
$$\Rightarrow \theta = n\pi/4 + (-1)n(-\pi/24)$$

$$\therefore \theta = 7\pi/24, 11\pi/24.$$

5. Continuity and Differentiability

1. (a) $f(x) = \frac{x^2 - (A+2)x + A}{x-2}$; $x \neq 2$

$$= 2 \quad ; \quad x = 2$$



$$\begin{aligned} L.H.L &= \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2-h) \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 - (A+2)(2-h) + 2A}{2-h-2} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 - (A+2)(2-h) + 2A}{-h} \\ &= \lim_{h \rightarrow 0} \frac{2(2-h)(-1) - (A+2)(-1) + 0}{-1} \\ &= \lim_{h \rightarrow 0} \frac{2(2-h) - (A+2)}{1} \\ &= 4 - (A+2) \\ &= 2 - A \end{aligned}$$

Since, $f(x)$ is continuous (There is no need to calculate both limits)

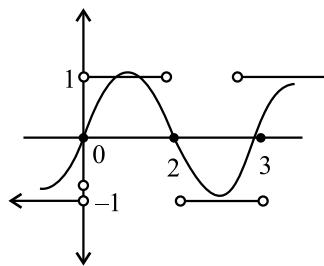
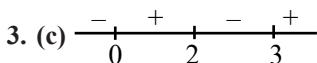
Hence, $L.H.L = f(2)$

$$2 - A = 2$$

∴ Option (a) is correct Answer.

2. (c) $f(x) = \begin{cases} \frac{36^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}, & x \neq 0 \\ k, & x = 0 \end{cases}$

$$\begin{aligned} &= \frac{9^x \cdot 4^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1+\cos x}} = \frac{9^x(4^x - 1) - 1(4^x - 1)}{\sqrt{2} - \sqrt{1+\cos x}} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{(9^x - 1)(4^x - 1)}{x^2(2 - 1 - \cos x)} \times (\sqrt{2} + \sqrt{1+\cos x}) \times x^2 \\ &= \lim_{x \rightarrow 0} \frac{\ln 9 \times \ln 4}{1 - \cos x} (\sqrt{2} + \sqrt{1+\cos x}) \\ &= 2 \ln 9 \times \ln 4 (\sqrt{2} + \sqrt{2}) \\ &= 16 \sqrt{2} \ln 3 \ln 2. \end{aligned}$$



$$f(x) = \operatorname{sgn}(x), g(x) = x(x-3)(x-2)$$

$$f(g(x)) = 0 \text{ at } x = 0, 3, 2$$

4. (c)

5. (d) $f(x) = \lim_{x \rightarrow \infty} \sin^{2n} x = \lim_{x \rightarrow \infty} (\sin^2 x)^n$

$$= \begin{cases} 1, & x = (2n+1)\frac{\pi}{2}, n \in I \\ 0, & x \neq (2n+1)\frac{\pi}{2}, n \in I \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = (2n+1)\frac{\pi}{2}$, $n \in I$.

6. (c) $f(0) = 0$

$$\text{For } \lim_{x \rightarrow 0} f(x) = 0 \therefore \lim_{x \rightarrow 0} x p \sin \frac{1}{x} = 0.$$

This is possible only when $p > 0$... (i)

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^p \sin \frac{1}{h} - 0}{h} \\ &= \lim_{x \rightarrow 0} h^{p-1} \sin \frac{1}{h} \end{aligned}$$

$f'(0)$ will exist only when $p > 1$

∴ $f(x)$ will not be differentiable if $p \leq 1$... (ii)

From (i) and (ii), for $f(x)$ to be not differentiable but continuous at $x = 0$, possible values of p are given by $0 < p \leq 1$.

7. (b) $\lim_{x \rightarrow 0} \frac{\int_0^x f(u) du}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{1}$

$$\left[\begin{array}{l} \text{Using L'Hospital's Rule} \\ \because f(x) \text{ is continuous at } x = 0 \end{array} \right] = f(0) = 2.$$

8. (c) Since $g(x)$ is the inverse of $f(x)$, therefore $g(x) = f^{-1}(x) \Rightarrow f\{g(x)\} = x$

Differentiate both sides w.r.t. x

$$f'\{g(x)\} g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^3$$

9. (c) $\frac{f'(c)}{8'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)} = \frac{6-2}{2-0} = 2 \Rightarrow f'(c) = 2g'(c)$

10. (b) Given $f''(x) = 6(x-1)$

$$\Rightarrow f'(x) = \frac{6(z-1)^2}{2} + C$$

$$3 = 3 + c$$

$$c = 0$$

$$\left[\begin{array}{l} \because f(x) = y = 3x + 5 \\ f(x) = 3 \forall x \in R \end{array} \right]$$

$$\text{So } f'(x) = 3(x-1)^2$$

$$f(x) = (x-1)^3 + c_1 \text{ as curve passes through } (2, 1)$$

$$1 = (2-1)^3 + c_1 \Rightarrow c_1 = 0 \therefore f(x) = (x-1)^3$$

11. (d) $f(-1-0) = -1$, $f(-1) = -(-1) = 1$

$$\Rightarrow f(-1-0) \neq f(-1)$$

$\Rightarrow f(x)$ is not continuous at $x = -1$

Further, $f(1) = -1$

$$f(1+0) = 1 \quad \Rightarrow f(1) \neq f(1+0)$$

$\Rightarrow f(x)$ is not continuous at $x = 1$.

12. (b) Since $f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

but $f(0) = 0$ (given)

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^k \cos(1/x)$$

$= 0$, if $k > 0$.

13. (b) Obviously function $f(x)$ is discontinuous at $x = 0$ and $x = 1$ because the function is not defined, when $x < 0$ and $x > 1$, therefore $f(0-0)$ and $f(1+0)$ do not exist. Again

$$f\left(\frac{1}{2} + 0\right) = \lim_{x \rightarrow 1/2} \left(\frac{3}{2} - x\right) = 1$$

$$f\left(\frac{1}{2} - 0\right) = \lim_{x \rightarrow 1/2} \left(\frac{1}{2} - x\right) = 0$$

$$\therefore f\left(\frac{1}{2} + 0\right) \neq f\left(\frac{1}{2} - 0\right)$$

function $f(x)$ is discontinuous at $x = \frac{1}{2}$

14. (c) $\because f(x)$ is continuous at $x = 2$

$$\therefore f(2-0) = f(2+0) = f(2) = k$$

But $f(2+0)$

$$= \lim_{h \rightarrow 0} \frac{(2+h)^3 + (2+h)^2 - 16(2+h) + 20}{(2+h-2)^2}$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + 7h^2}{h^2} = 7$$

15. (b) Since $f(x)$ is continuous at $x = 2$

$$\therefore f(2) = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow 1 = \lim_{x \rightarrow 2^+} (ax+b)$$

$$\therefore 1 = 2a + b \quad \dots(1)$$

Again $f(x)$ is continuous at $x = 4$,

$$\Rightarrow f(4) = \lim_{x \rightarrow 4^+} f(x)$$

$$\Rightarrow 7 = \lim_{x \rightarrow 4} (ax+b)$$

$$\therefore 7 = 4a + b \quad \dots(2)$$

Solving (1) and (2), we get $a = 3$, $b = -5$.

16. (b) Let us first examine continuity at $x = 0$.

$$f(0) = 0 \quad (\because 0 \in Q)$$

$$= f(0-0) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \{-h \text{ or } h \text{ according as } -h \in Q \text{ or } -h \notin Q\}$$

$$= 0$$

$$f(0+0) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \{h \text{ or } -h\} = 0$$

$$f(0) = f(0-0) = f(0+0)$$

$\Rightarrow f(x)$ is continuous at $x = 0$.

Now let $a \in R$, $a \neq 0$, then

$$f(a-0) = \lim_{h \rightarrow 0} f(a-h)$$

$$= \lim_{h \rightarrow 0} \{(a-h) \text{ or } -(a-h)\}$$

$$= a \text{ or } -a, \text{ which is not unique.}$$

$\Rightarrow f(a-0)$ does not exist

$\Rightarrow f(x)$ is not continuous at $a \in R_0$.

Hence $f(x)$ is continuous only at $x = 0$.

17. (d) We know that $[x]$ is discontinuous at every integer. Therefore it is continuous only at $x = 1/2$, while the function x is continuous at all points $x = 0, -1, 1, 1/2$. Thus the given function is continuous only at $x = 1/2$.

18. (b) at $x = 0$:

$$f'(0-0) = \lim_{h \rightarrow 0} \frac{|0-h|-0}{-h} = -1$$

$$f'(0+0) = \lim_{h \rightarrow 0} \frac{|0+h|-0}{h} = 1$$

Now, since $f'(0-0) \neq f'(0+0)$

$\Rightarrow f(x)$ is not differentiable at $x = 0$.

19. (b) Differentiability at $x = 0$

$$R[f'(0)] = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(0+h)^2 - 0}{h} \lim_{h \rightarrow 0} h = 0$$

$$L[f'(0)] = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$\lim_{h \rightarrow 0} \frac{-(0-h)-0}{-h} = -1$$

$$\therefore R[f'(0)] \neq L[f'(0)]$$

$\therefore f(x)$ is not differentiable at $x = 0$.

Differentiability at $x = 1$

$$R[f'(1)] = \lim_{h \rightarrow 0} \frac{f(1+h)^3 - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^3 - (1+h)+1-1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h+3h^2+h^3}{h} = 2$$

$$L[f'(1)] = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$\lim_{h \rightarrow 0} \frac{(1-h)-1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h + h^2}{-h} = 2$$

Thus $R[f'(1)] = L[f'(1)]$

\therefore function $f(x)$ is differentiable at $x = 1$

20. (b) When $x \neq 0$

$$\begin{aligned} f'(x) &= 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right) \\ &= 2x \sin \frac{1}{x} - \cos \left(\frac{1}{x} \right) \end{aligned}$$

which exists finitely for all $x \neq 0$

$$\text{and } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin 1/x}{x} = 0$$

$\therefore f$ is also derivable at $x = 0$. Thus

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned} \text{Also } \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right) \\ &= 2 - \lim_{x \rightarrow 0} \cos \frac{1}{x} \end{aligned}$$

But $\lim_{x \rightarrow 0} \cos \frac{1}{x}$ does not exist, so $\lim_{x \rightarrow 0} f'(x)$ does not exist. Hence f' is not continuous (so not derivable) at $x = 0$.

6. Application of Derivatives

1. (d)

2. (b) The equation of the line joining the points $(0, 3)$ and $(5, -2)$ is

$$y = -x + 3$$

The line intersects the curve $y = \frac{ax}{x+1}$ at points whose x-coordinates are given by

$$-x + 3 = \frac{ax}{x+1}$$

$$\Rightarrow (-x + 3)(x + 1) = ax$$

$$\Rightarrow -x^2 + 2x + 3 = ax$$

$$\Rightarrow x^2 + (a - 2)x + 3 = 0$$

The line will be tangent to the curve if the quadratic equation above has a double root, that is when its discriminant equal 0

$$\text{So, } (a - 2)^2 - 4 \times 1 \times (-2) = 0$$

$$\Rightarrow (a - 2)^2 + 12 = 0$$

This is impossible because $(a - 2)^2 + 12 > 0$ for all real number a.

3. (d) $f''(x) < 0 \forall x \in R$

$\Rightarrow f'(x)$ is a decreasing function.

$$g(x) = f(x^2 - 2) + f(6 - x^2)$$

$$g'(x) = 2x [f'(x^2 - 2) - f'(6 - x^2)]$$

$$g'(x) > 0$$

$$\Rightarrow f'(x^2 - 2) > f'(6 - x^2)$$

$$\Rightarrow x^2 - 2 < 6 - x^2$$

$$\Rightarrow x < 2$$

$$g'(x) < 0$$

$$\Rightarrow f'(x^2 - 2) < f'(6 - x^2)$$

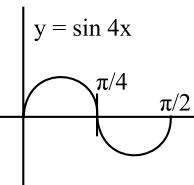
$$\Rightarrow x^2 - 2 > 6 - x^2$$

$$\Rightarrow x > 2$$

So, $g(x)$ has a local maxima at $x = 2$.

4. (c) $f(x) = \sin^4 x + \cos^4 x$

$$\begin{aligned} f'(x) &= 4 \sin^3 x \cos x + 4 \cos^3 x (-\sin x) \\ &= 4 \sin x \cos x (\sin^2 x - \cos^2 x) \\ &= -2 \sin 2x \cos 2x = -\sin 4x \end{aligned}$$



Since, $f(x)$ is increasing when $f'(x) > 0$

$$\Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0 \Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

5. (c) $f(x) = \lambda x^3 - 9x^2 + 9x + 10$

$$f'(x) = 3\lambda x^2 - 18x + 9$$

$$f'(x) > 0$$

$$\Rightarrow 3\lambda x^2 - 18x + 9 > 0$$

$$\Rightarrow \lambda x^2 - 6x + 3 > 0$$

$$D = 18 - 6\lambda = 0$$

$$\Rightarrow \lambda = 3$$

So, $f(x)$ is increasing on R for $\lambda \geq 3$.

6. (d) Let $BD_1 = x \Rightarrow BC_1 = (a - x)$

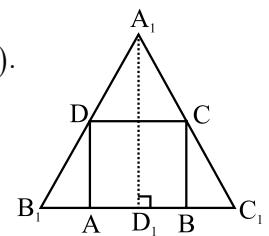
$$\Rightarrow BC = (a - x) \tan \frac{\pi}{3} = \sqrt{3}(a - x).$$

Now, area of rectangle ABCD,

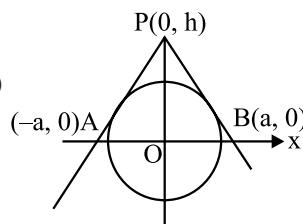
$$\Delta = (AB)(BC) = 2\sqrt{3x}(a - x).$$

$$\Rightarrow \Delta \leq 2\sqrt{3} \left(\frac{x + a - x}{2} \right)^2 = \frac{\sqrt{3}a^2}{2}$$

(using A.M. \geq G.M.)



7. (d)



Let AP and BP be the tangents and B be $(a, 0)$ then A is the point $(-a, 0)$

Equation of BP is: $hx + ay = ah$;

BP is a tangent to $x^2 + y^2 = 16$

$$\Rightarrow a^2 h^2 = 16(h^2 + a^2) \Rightarrow a = \pm \frac{4h}{\sqrt{h^2 - 16}}$$

$$A = \text{Area} = \frac{4h}{\sqrt{h^2 - 16}} \times h = \frac{4h^2}{\sqrt{h^2 - 16}}$$

$$A \text{ is minimum} \Rightarrow \frac{dA}{dh} = 0$$

$$\Rightarrow \frac{8h\sqrt{h^2 - 16} - 4h^2}{\sqrt{h^2 - 16}} \frac{2h}{2\sqrt{h^2 - 16}} = 0$$

On solving, we get $h = 4\sqrt{2}$

8. (a) We know that, $v^2 - u^2 = 2ag$

$$\Rightarrow 0 - 48^2 = 2(-32)h$$

$$\Rightarrow h = \frac{2304}{64} = 36 \text{ m}$$

\therefore the greatest height = $64 + 36 = 100$ meters

9. (c) $f'(x) = 0$ for $x \leq 1$

$$f'(x) = -1 < 0 \text{ for } x > 1$$

So, neither a point of local minima nor maxima.

10. (c) Let r, V and S be radius, volume and surface area of the sphere at time t.

$$V = \frac{4}{3} \pi r^3 \text{ and } S = 4\pi r^2$$

$$\text{Rate of change of volume} = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dv}{dt} = 5 \Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$$

Rate of change of surface area = $\frac{ds}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \frac{3}{4\pi r^2} = \frac{6}{r}$
 $\frac{ds}{dt} = 5 \Rightarrow r = \frac{6}{5} \text{ m}$

11. (b) Given curve is $y = a^{1-n} \cdot x^n$

Taking logarithm of both sides, we get, $\ln y = (1 - n) \ln a + n \ln x$

Differentiating both sides w.r.t x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 0 + \frac{n}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{ny}{x} \quad \dots(1)$$

Lengths of sub-normal = $y \frac{dy}{dx} = y \cdot n \frac{y}{x} \dots [\text{from 1}]$

$$= \frac{ny^2}{x} = n \cdot \frac{(a^{1-n} x^n)^2}{x}$$

$$(\because y = a^{1-n} \cdot x^n) = n \cdot a^{2-2n} \cdot x^{2n-1}$$

Since lengths of sun-normal is to be constant, so x should not appear in its value i.e., $2n - 1 = 0$. $n = \frac{1}{2}$

12. (d)

13. (d) We have, $y \sec(\tan^{-1}x)$

$$\frac{dy}{dx} = \sec(\tan^{-1}x) \cdot \tan(\tan^{-1}x) \cdot \frac{1}{1+x^2}$$

$$\frac{dy}{dx}_{x=1} = \sqrt{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}$$

14. (d)

15. (c)

16. (b) For the range of the expression

$$\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7} = y = \frac{ax^2 + bx + c}{px^2 + qx + r},$$

[find the solution of the inequality $A y^2 + B y + K \geq 0$]

where $A = q^2 - 4pr = -3$, $B = 4ar + 4pc - 2bq = 126$

$K = b^2 - 4ac = -123$ i.e. solve $= 3y^2 + 126 + y - 123 \geq 0$

$$\Rightarrow 3y^2 + 126y + 123 \leq 0$$

$$\Rightarrow y^2 + 42y + 41 \leq 0$$

$$\Rightarrow (y + 1)(y + 41) \leq 0$$

$$\Rightarrow -1 \leq y \leq -41$$

Maximum value of y is 41

17. (d)

18. (b)

19. (b) $f'(x) = 6x^2 - 42x + 36$

$$f''(x) = 12x - 42$$

$$\text{Now } f'(x) = 0 \Rightarrow 6(x^2 - 7x + 6) = 0$$

$$\Rightarrow x = 1, 6$$

$$\text{Also } f''(1) = 12 - 42 = -30 < 0$$

$\therefore f(x)$ has a maxima at $x = 1$

20. (b) Let $A = x + y = x + 1/x$ ($\because xy = 1$)

$$\Rightarrow \frac{dA}{dx} = 1 - \frac{1}{x^2}, \frac{d^2A}{dx^2} = \frac{2}{x^3}$$

$$\text{Now } \frac{dA}{dx} = 0 \Rightarrow x = 1, -1$$

$$\text{Also at } x = 1, \frac{d^2A}{dx^2} = 2 > 0$$

$x = 1$ is a minimum point of A. So minimum value of $A = 1 + 1/1 = 2$.

7. Integral

1. (c) Here $I = \int \frac{1-\cos x}{2} dx$

$$= \frac{1}{2}(x - \sin x) + c$$

2. (c) $\int \frac{5x+7}{x} dx = \int \left(\frac{5x}{x} + \frac{7}{x} \right) dx$

$$= \int 5 dx + \int \frac{7}{x} dx = 5 \int 1 dx + 7 \int \frac{1}{x} dx$$

$$= 5x + 7 \log x + c$$

3. (b) $\int \left(x - \frac{1}{x} \right)^3 dx$

$$= \int \left(x^3 - 3x^2 \cdot \frac{1}{x} + 3x \cdot \frac{1}{x^2} - \frac{1}{x^3} \right) Edx$$

$$[\because (a-b)^3 = (a^3 - 3a^2b + 3ab^2 - b^3)]$$

$$= \int \left(x^3 - 3x + \frac{3}{x} - \frac{1}{x^3} \right) dx$$

$$= \int x^3 dx - 3 \int x dx + 3 \int \frac{1}{x} dx - \int \frac{1}{x^3} dx$$

$$= \frac{x^{3+1}}{3+1} - 3 \cdot \frac{x^{1+1}}{1+1} + 3 \log x - \frac{x^{-3+1}}{-3+1} + c$$

$$= \frac{x^4}{4} - \frac{3}{2}x^2 + 3 \log x + \frac{1}{2x^2} + c$$

4. (a) $I = \int (\tan^2 x + \cot^2 x + 2) dx$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$= \tan x - \cot x + c$$

5. (a) $\int \frac{x^2 - 1 + 1}{x^2 - 1} dx$

$$= \int \left(1 + \frac{1}{x^2 - 1} \right) dx$$

$$= x + \frac{1}{2} \log \left(\frac{x-1}{x+1} \right) + c$$

$$= x + \log \sqrt{\frac{x-1}{x+1}} + c$$

6. (d) $\int \frac{1}{x \log x} dx = \int \frac{1}{x} \cdot \frac{1}{\log x} dx$

$$\text{put } \log x = t, \frac{1}{x} dx = dt$$

$$\therefore \int \frac{1}{x} \cdot \frac{1}{\log x} dx = \int \frac{1}{t} dt$$

$$\therefore \int \frac{1}{t} dt = \log t + c = \log (\log x) + c$$

(putting the value of $t = \log x$)

7. (c) $\int \frac{\sin 2x}{1 + \cos^4 x} dx$

$$= \int \frac{2 \sin x \cos x}{1 + \cos^4 x} dx$$

$$= \int \frac{2 \cos x}{1 + \cos^4} \cdot \sin x dx$$

$$= \int \frac{2t}{1+t^4} \cdot (-dt)$$

$$= \int \frac{2t}{1+(t^2)^2} dt$$

$$= \int \frac{1}{1+u^2} du$$

$$= -\arctan(u)$$

$$= -\arctan(t^2)$$

$$= -\arctan(\cos^2 x) + c$$

8. (c) $\int \frac{be^x}{\sqrt{a+be^x}} dx$, putting $a+be^x = t$

$$be^x dx = dt$$

$$\therefore \int \frac{be^x}{\sqrt{a+be^x}} dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c$$

$$= 2\sqrt{a+be^x} + c$$

9. (b) $I = \int \sqrt{\frac{1+\cos x}{1-\cos x}} dx$

$$= \int \sqrt{\frac{2\cos^2(x/2)}{2\sin^2(x/2)}} dx$$

$$= \int \cot\left(\frac{x}{2}\right) dx$$

$$= 2 \log \sin\left(\frac{x}{2}\right) + c$$

10. (b) $I = \int \frac{\sqrt{\tan x}}{\tan x} \sec^2 x dx$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = 2\sqrt{\tan x} + c$$

11. (a) $\int \sin^5 x \cdot \cos^3 x dx$

Assumed that $\sin x = t$

$$\therefore \cos x dx = dt$$

$$\int t^5 (1-t^2) dt = \int (t^5 - t^7) dt$$

$$= \frac{t^6}{6} - \frac{t^8}{8} + c$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$$

12. (c) $\int \log x \, dx = \int \log x \cdot 1 \, dx$

[Integrating by parts, taking $\log x$ as first part and 1 as second part]

$$= (\log x) \cdot x - \int \left\{ \frac{d(\log x)}{dx} \right\} \cdot x \, dx$$

$$= x \log x - \int \frac{1}{x} \cdot x \, dx = (x \log x - x) + c$$

$$= x (\log x - 1) + c = \log \left(\frac{x}{e} \right) + c$$

13. (a) $\int_0^1 \frac{x^3}{1+x^8} \, dx = \frac{1}{4} \left[\tan^{-1}(x) \right]_0^1 = \frac{\pi}{16}$

14. (d)

15. (a) Let $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos(\pi-x)} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1-\cos x}$

On adding, we have, $2I \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{1-\cos^2 x} \, dx$

$$\Rightarrow I \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos ec^2 x \, dx = -\cot x \Big|_{\pi/4}^{3\pi/4} = 2$$

16. (b) It is fundamental property.

$$= \int \frac{1}{x} \, dx - \int \frac{1-\sin x}{x+\cos x} \, dx$$

$$= \log x - \log(x + \cos x) + c$$

$$= \log \left(\frac{x}{x + \cos x} \right) + c$$

17. (b) $f(2-x) = f(2+x)$

\Rightarrow Function is symmetrical about $x = 2$

and $f(4-x) = f(4+x) \Rightarrow$ Function is symmetrical about $x = 4$

$\Rightarrow f(x)$ is periodic with period 2.

$$\int_{10}^{50} f(x) \, dx = \int_{2(5)}^{2(25)} f(x) \, dx = (25-5) \int_0^2 f(x) \, dx = 20 \times 5 = 100$$

18. (c) $f(x) = e^{\cos^2 x} \cos^3(2n+1)x$

$$f(\pi-x) = e^{\cos^2(\pi-x)} \cos^3(2n+1)(\pi-x)$$

$$= -e^{\cos^2 x} \cos^3(2n+1)x$$

$$= -f(x)$$

$$\Rightarrow \int_0^\pi f(x) = e^{\cos^2 x} \cos^3(2n+1)x = 0$$

19. (b) $f(x) = \int_0^x \sqrt{t} \sin t \, dt$

$$f'(x) = \sqrt{x} \sin x$$

$$f''(x) = \sqrt{x} \cos x + \frac{1}{2} x^{-1/2} \sin x$$

$$f''(\pi) = -\sqrt{\pi} < 0; f''(2\pi) = \sqrt{2\pi} > 0$$

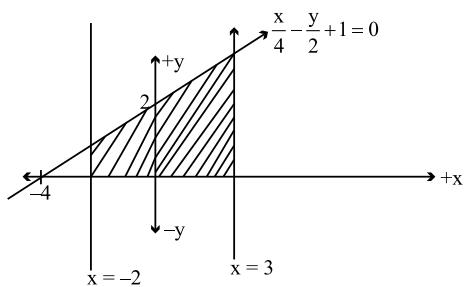
Thus maximum at π and minimum at 2π .

20. (c) $\lim_{x \rightarrow 2} \int_6^{f(x)} \frac{4t^3}{x-2} \, dt = \lim_{x \rightarrow 2} \frac{f'(x) \times 4(f(2))^3}{1}$

$$= 4f'(2) \times (f(2))^3 = \frac{1}{48} \times 4 \times 6 \times 6 \times 6 = 18$$

8. Application of Integrals

1. (a)



$$\text{Area} = \int y dx$$

$$\begin{aligned} &= \int_{x=-2}^{x=3} \left(\frac{x}{2} + 2 \right) dx \\ &= \int_{x=-2}^{x=3} \frac{x}{2} dx + \int_{x=-2}^{x=3} 2 dx \\ &= \frac{1}{2} \left[\frac{x^2}{2} \right]_{-2}^3 + 2[x]_{-2}^3 \\ &= \frac{1}{4} [9 - 4] + [3 - (-2)] \\ &= \frac{45}{4} \end{aligned}$$

2. (b) Area b/w $y = \tan x$, x-axis & $x = \frac{\pi}{3}$

$$\begin{aligned} A &= \int_{x_1}^{x_2} y dx \\ A &= \int_{x=0}^{\pi/3} \tan x dx \\ &= [\log(\cos x)]_0^{\pi/3} \\ &= [\log(\cos x)]_{\pi/3}^0 \\ &= \log(\cos(0)) - \log(1/2) \\ &= \log 1 - \log(1/2) \\ &= \log 2. \end{aligned}$$

3. (c) $y = \log_e x$, x axis & $x = e$

$$\begin{aligned} A &= \int_{x_1}^{x_2} y dx \\ A &= \int_1^e \log_e x dx \end{aligned}$$

Integration by parts

$$\begin{aligned} A &= \log_e x \int dx - \int \left(\frac{1}{x} \times \int dx \right) dx \\ &= [x \log_e x - x]_1^e \\ &= 1 \end{aligned}$$

4. (a) $y = \frac{1}{\cos^2 x}$

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{\cos^2 x} dx \\ &= \int_0^{\pi/4} \sec^2 x dx \\ &= [\tan x]_0^{\pi/4} = 1 \end{aligned}$$

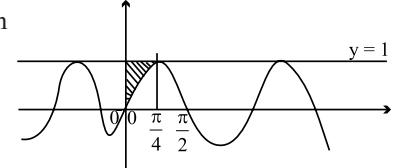
5. (a) $y^2 = 4x$, y-axis & $Y = -1$ & $y = 3$

$$\begin{aligned} A &= \int_{y_1}^{y_2} x dy \\ &= \int_{-1}^3 \frac{y^2}{4} dy \\ &= \frac{1}{4} \left[\frac{y^2}{4} \right]_{-1}^3 \\ &= \frac{1}{12} [3^3 - (-1)^3] = \frac{1}{12} [27 + 1] = \frac{28}{12} = \frac{7}{3} \end{aligned}$$

6. (d) $A = \int dy x$

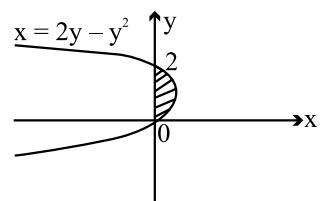
But here, shaded region
area will be

$$\begin{aligned} A &= \int_0^{\pi/4} (1 - \sin 2x) dx \\ &= \left[x + \frac{\cos 2x}{2} \right]_0^{\pi/4} = \frac{\pi}{4} + 0 - \frac{1}{2} \end{aligned}$$



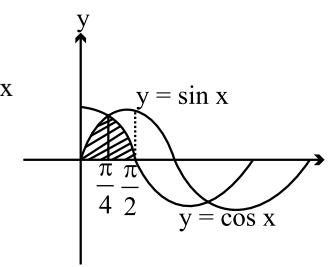
7. (b) $x = 2y - y^2$ and y-axis

$$\begin{aligned} A &= \int_0^2 x dy \\ &= \int_0^2 (2y - y^2) dy \\ &= \left[2 \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3} \end{aligned}$$

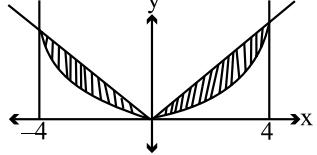


8. (d) Required Area

$$\begin{aligned} &= \int_0^{\pi/4} \sin x dx + \int_{4/\pi}^{\pi/2} \cos x dx \\ &= [-\cos x]_0^{\pi/4} + [\sin x]_{4/\pi}^{\pi/2} \\ &= \frac{-1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}} \\ &= 2 - \sqrt{2} \end{aligned}$$



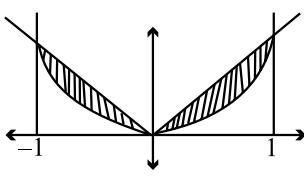
9. (d) Area b/w $x^2 = 4y$ & $y = |x|$



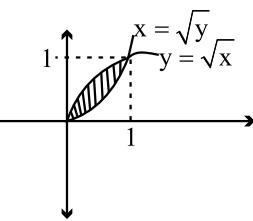
$$\begin{aligned} A &= \int_{-4}^0 \left(|x| - \frac{x^2}{4} \right) dx + \int_0^4 \left(|x| - \frac{x^2}{4} \right) dx \\ &= 2 \times \int_0^4 \left(|x| - \frac{x^2}{4} \right) dx \\ &= 2 \times \left[8 - \frac{16}{3} \right] \\ &= \frac{16}{3} \end{aligned}$$

10. (d) $A = 2x \int_0^1 (x - x^2) dx$

$$\begin{aligned} &= 2 \times \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\ &= 2 \times \left[\frac{1}{2} - \frac{1}{3} \right] \\ &= 2 \times \frac{1}{6} = \frac{1}{3} \end{aligned}$$

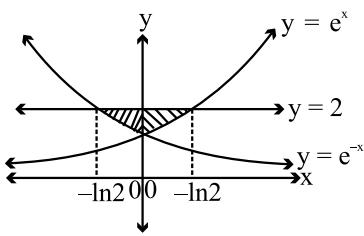


11. (c)



$$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x^2) dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \end{aligned}$$

12. (b)

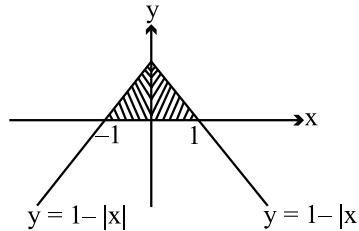


$$\begin{aligned} \text{Area} &= 2 \times \int_0^{\ln 2} (2 - e^x) dx \\ &= 2 \times [2x - e^x]_0^{\ln 2} \\ &= 2 \times [2\ln 2 - 2 + 1] = 2[2\ln 2 - 1] \end{aligned}$$

$$= 4 \ln 2 - 2$$

$$= 2 \left[\ln \frac{4}{e} \right]$$

13. (a)

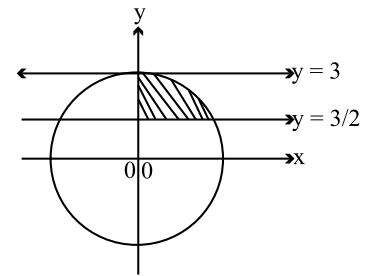


$$\begin{aligned} A &= \int_0^1 (1-x) dx = 2 \left[x \frac{x^2}{2} \right]_0^1 \\ &= 2 \left[1 - \frac{1}{2} \right] = 1 \end{aligned}$$

14. (d) $A = \int_{y_1}^{y_2} x dx$

$$= \int_{3/2}^3 (9-y)^{1/2} dy$$

Let $y = 3 \sin \theta$



$$dy = 3 \cos \theta d\theta$$

$$\text{So, } A = \int_{\pi/6}^{\pi/2} (9 - 9 \sin^2 \theta)^{1/2} \times 3 \cos \theta d\theta$$

$$= \int_{\pi/6}^{\pi/2} 9(1 - \sin^2 \theta)^{1/2} \cos \theta d\theta$$

$$= 9 \int_{\pi/6}^{\pi/2} \cos^2 \theta d\theta$$

$$= 9 \int_{\pi/6}^{\pi/2} \left(\frac{\cos 2\theta + 1}{2} \right) d\theta$$

$$= \frac{9}{2} \left[\frac{\sin 2\theta}{2} + \theta \right]_{\pi/6}^{\pi/2}$$

$$= \frac{9}{2} \left[\frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} \right]$$

$$= \frac{3}{8} [4\pi - 3\sqrt{3}]$$

$$15. (c) A = \int_1^b f(x) dx = (b-1) \sin(3b+4)$$

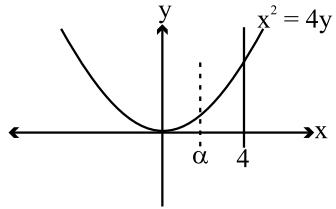
diff w.r.t 'b' both sides

$$\Rightarrow f(b) - 0 = 3(b-1) \cos(3b+4) + \sin(3b+4)$$

replacing 'b' by 'x' we get

$$f(x) = 3(x-1) \cos(3x+4) + \sin(3x+4)$$

16. (d)



According to Question

$$\int_0^4 \frac{x^3}{x} dx = 2 \int_0^{\alpha} \frac{x^2}{4} dx$$

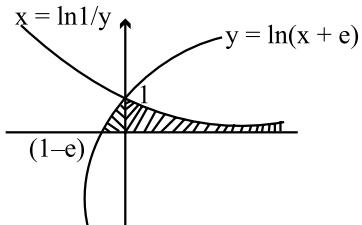
$$\Rightarrow \left[\frac{x^3}{12} \right]_0^4 = 2 \left[\frac{x^3}{12} \right]_0^{\alpha}$$

$$\Rightarrow \frac{64}{12} = 2 \frac{\alpha^3}{12}$$

$$\Rightarrow \alpha^3 = 32$$

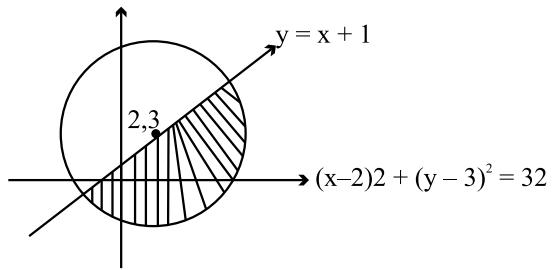
$$\Rightarrow \alpha = 2^{5/3}$$

17.

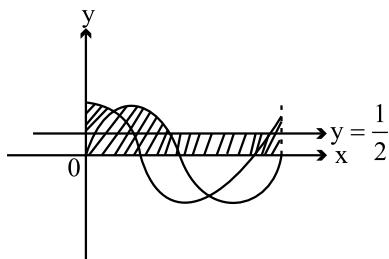


There is no enclosed area.

18. (c) Hint:-



19. (d)



20. (a) Refer Q. 10

9. Differential Equations

1. (a) On dividing the given equation by $\sin x$,

$$2y \frac{dy}{dx} + y^2 \cot x = 2 \cos x$$

Now put $y^2 = v$, then $2y \frac{dy}{dx} = \frac{dv}{dx}$

Then the given equation becomes

$$\frac{dv}{dx} + v \cot x = 2 \cos x$$

I.E. $= e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Solution is $v \cdot \sin x = \int \sin x \cdot (2 \cos x) dx + c$

$$\Rightarrow y^2 \sin x = \sin^2 x + c$$

When $x = \frac{\pi}{2}$, $y = 1$, then $c = 0$

$$\Rightarrow y^2 = \sin x.$$

2. (b) $\frac{dy}{dx} + (3x)y = x$

I.F. $= e^{\int 3x dx} = e^{\frac{3x^2}{2}}$

Solution of given equation

$$ye^{\frac{3x^2}{2}} = \int x \cdot e^{\frac{3x^2}{2}} dx + c = \frac{1}{3} e^{\frac{3x^2}{2}} + c$$

If curve passes through $(0, 4)$, then

$$4 - \frac{1}{3} = c \text{ and } c = \frac{11}{3}$$

$$y = \frac{1}{3} + \frac{11}{3} e^{-\frac{3x^2}{2}} \quad \text{so } 3y = 1 + 11 e^{-\frac{3x^2}{2}}$$

3. (d) We have $\frac{dy}{dx} = -\frac{(y+1)\cos x}{2+\sin x}$

$$\int \frac{dy}{y+1} = \int \frac{\cos x}{2+\sin x} dx$$

$$\Rightarrow \ln(y+1) = -\ln(2+\sin x) + \ln \lambda \Rightarrow (y+1)(2+\sin x) = \lambda$$

As $y(0) = 1 \Rightarrow 2.2 = \lambda$ or $\lambda = 4$

$$\text{At } x = \frac{\pi}{2}, y\left(\frac{\pi}{2}\right) = \frac{4}{2+1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

4. (b) $y = a + be^{5x} + ce^{-7x}$

$$y_1 = 0 + 5be^{5x} - 7ce^{-7x}$$

dividing by ye^{5x}

we get

$$e^{-5x} y_1 = 5b - 7ce^{-12x}$$

again differentiating on both sides wrt x

$$e^{-5x} y_2 + y_1(-5)e^{-5x} = 0 + 84ce^{-12x}$$

dividing by e^{-12x} , then we get

$$e^{7x}(y_2 - 5y_1) = 84c$$

differentiating wrt x

$$e^{7x}(y_3 - 5y_2) + (y_2 - 5y_1)7e^{7x} = 0$$

$$\text{or } y_3 + 2y_2 - 35y_1 = 0$$

5. (c) $e^2y dy = dx$

Hence, $\frac{e^{2y}}{2} + c = x$

Then, $c = 5 - \frac{1}{2} = \frac{9}{2}$

At $y = 3$, the value of x is $\frac{e^6 + 9}{2}$

6. (c)

7. (d) We have, $ydx - (x + 3y^2) dy = 0$

$$\Rightarrow ydx = (x + 3y^2) dy \Rightarrow \frac{dx}{dy} = \frac{x}{y} + 3y \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

This is homogeneous linear differential equation.

$$\therefore \text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$\therefore \text{Solution is, } \frac{x}{y} = \int 3y \cdot \frac{1}{y} dy \Rightarrow \frac{x}{y} = 3y + c \quad \dots(i)$$

Since (i) passes through $(1, 1)$ $\therefore 1 = 3 + c \Rightarrow c = -2$

\therefore Required curve is $x = 3y^2 - 2y$

This curve also passes through the point $\left(-\frac{1}{3}, \frac{1}{3}\right)$

8. (b) $(x+1) \frac{dy}{dx} + 1 = e^{x-y}$

$$\Rightarrow (x+1) \frac{dy}{dx} = e^{x-y} - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{x-y} - 1}{x+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{x-y}}{x+1} - \frac{1}{x+1}$$

On integrating we get

$$e^y (x+1) = e^x + c$$

9. (b) The given equation is $\frac{dx}{dt} = x + 1$. It may be written as

$$\frac{dx}{(x+1)} = dt$$

$$\Rightarrow \log(x+1) = t + c \quad \dots(1)$$

Initially, when $t = 0, x = 0$

$$\Rightarrow c = 0$$

$$\Rightarrow \log(x+1) = t$$

When $x = 99$, then $t = \log_e(100) = 2 \log_e 10$.

10. (d) $\frac{dp}{dt} = \frac{1}{2} p(t) - 200$

$$\Rightarrow \frac{dp}{p-400} = \frac{1}{2} dt$$

Integrating, we get, $\ln |p-400| = \frac{1}{2} t + c$

$$t = 0, p = 100 \Rightarrow \lambda \ln 300 = c$$

$$\text{Again, } \ln\left(\frac{p-400}{300}\right) = \frac{t}{2} \Rightarrow |p-400| = 300e^{t/2}$$

$$\therefore 400-p = 300e^{t/2} \quad (p < 400) \quad \therefore p = 400 - 300e^{t/2}$$

11. (c) $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200 \quad \dots\dots(1)$

$$\frac{\frac{dp(t)}{dt}}{\frac{p(t)}{2} - 200} = dt$$

now integrating on the both sides:

$$2 \log\left[\frac{p(t)}{2} - 200\right] = t + C \text{ where } C \text{ is an integration constant.}$$

$$\log\left[\frac{p(t)}{2} - 200\right] = \frac{t+C}{2}$$

$$\frac{p(t)}{2} - 200 = e^{t/2} \cdot k \text{ where } k = e^{C/2} = \text{constant}$$

$$p(t) - 400 = 2 \cdot e^{t/2} \cdot k$$

$$p(t) = 400 + 2e^{t/2} \cdot k \quad \dots\dots(2)$$

$$\text{given : } p(0) = 100$$

$$100 = 400 + 2 \cdot k$$

$$\Rightarrow k = -150$$

thus the required equation is:

$$p(t) = 400 - 300 \cdot e^{t/2}$$

12. (a)

13. (c) Let the point on the curve is (x, y)

$$\text{the slope at } (x, y) = \frac{dy}{dx}$$

and according to question

$$\frac{dy}{dx} = 3x$$

$$\Rightarrow dy = 3x \, dx$$

integrating both sides

$$\int dy = \int 3x \, dx$$

$$\Rightarrow y = 3\left(\frac{x^2}{2}\right) + c \quad \dots\dots(1)$$

As the curve passes through point $(-1, -3)$ it will satisfy eq. (1)

$$-3 = \frac{3(1)^2}{2} + c$$

$$\Rightarrow -3 = \frac{3}{2} + c$$

$$\Rightarrow c = -3 - \frac{3}{2}$$

$$= \frac{-9}{2}$$

Equation of curve

$$y = \frac{3}{2}(x^2) \frac{-9}{2}$$

$$y = \frac{3x^2 - 9}{2} \Rightarrow 2y = 3x^2 - 9$$

$$\Rightarrow 2y = 3(x^2 - 3)$$

14. (c) $y = c_1 e^{c_2 x}$

Differentiating w.r.t. x, we get $y' = c_1 c_2 e^{c_2 x} = c_2 y \quad \dots\dots(1)$

Again differentiating w.r.t. x, $y'' = c_2 y'$

$$\text{From (i) and (ii) upon division } \frac{y'}{y''} = \frac{y}{y'} \Rightarrow y''y = (y')^2$$

which is the desired differential equation of the family of curves.

15. (d)

16. (c) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$

$$\Rightarrow e^y dy = (e^x + x^2) dx$$

On Integrating we get

$$e^y + e^x + \frac{x^3}{3} = c$$

17. (b) $\cos^2 x \frac{dy}{dx} + y = \tan x \quad \dots\dots(1)$

$$\Rightarrow \frac{dy}{dx} + y \sec^2 x = \frac{\tan x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} + y \sec^2 x = \frac{\sin x}{\cos^3 x}$$

The above is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where}$$

$$P = \sec^2 x; Q = \frac{\sin x}{\cos^3 x}$$

$$\text{Now, IF} = e^{\int P dx} = e^{\int \sec^2 x dx} = e^{\tan x}$$

Now, solution of (1) is given by,

$$y \times \text{IF} = \int [Q \times \text{IF}] dx + C$$

$$\Rightarrow ye^{\tan x} = \int \frac{\sin x}{\cos^3 x} e^{\tan x} dx + C$$

$$\Rightarrow ye^{\tan x} = \int e^{\tan x} \cdot \tan x \cdot \sec^2 x dx + C \quad \dots\dots(2)$$

$$\text{Let } I = \int e^{\tan x} \cdot \tan x \cdot \sec^2 x dx$$

$$\text{put } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$\text{So, } I = \int e^t \cdot t dt$$

$$\Rightarrow I = t \times \int e^t dt - \left[\frac{d}{dt}(t) \times \int e^t dt \right] dt$$

$$\Rightarrow I = te^t - \int e^t dt$$

$$\Rightarrow I = te^t - e^t$$

$$\Rightarrow I = \tan x \cdot e^{\tan x} - e^{\tan x}$$

Now, from (2), we get

$$ye^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + C$$

$$\Rightarrow ye^{\tan x} = e^{\tan x} (\tan x - 1) + C$$

$$\Rightarrow y = \tan x - 1 + Ce^{-\tan x}$$

18. (a) Evaporation rate \propto surface area

$$-\frac{dV}{dt} \propto A \Rightarrow -\frac{dV}{dt} = \lambda A$$

Substituting $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$, we have

$$-\frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right) = \lambda 4\pi r^2$$

$$\text{or } \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = -\lambda \times 4\pi r^2$$

$$\text{or } \frac{dr}{dt} = -\lambda$$

Integrating $\int dr = -\int \lambda dt$

$$r = -\lambda t + c$$

Boundary values at $t = 0$, $r = 3$ mm, and $t = 1$ hr, $r = 2$ mm.

From which $c = 3$, $\lambda = t$

$$r = 3 - t$$

19. (c) $f''(x) = g''(x)$

$$\Rightarrow f'(x) = g'(x) + c \dots\dots (1)$$

$$\Rightarrow f(x) = g(x) + cx + d \dots\dots (2)$$

Now $2f'(1) = 4 \Rightarrow f'(1) = 2$ and $g'(1) = 4$

$$\text{So, } 2 - 4 = c \Rightarrow c = -2 \quad [\text{by (1)}]$$

Substituting in equation (2) we get

$$f(x) = g(x) - 2x + d \dots\dots (3)$$

It is given that $f(2) = 3$ and $g(2) = 9$ so by (3) we have

$$f(2) = g(2) - 2 \times 2 + d \quad \text{at } x = 2$$

$$\Rightarrow d = -2$$

Substituting values of c and d in eqn. (2) and taking $x = 4$ we get

$$f(4) = g(4) - 2 \times 4 + -2$$

$$\Rightarrow f(4) - g(4) = -8 - 2 = -10$$

20. (b) $y dx = -(x^2 y + x) dy \Rightarrow ydx + xdy = -x^2 y dy$

$$\Rightarrow \frac{ydx + xdy}{(xy)^2} = \frac{-dy}{y} \Rightarrow \frac{d(xy)}{(xy)^2} = -\frac{dy}{y}$$

$$\Rightarrow d\left(-\frac{1}{xy}\right) = -\frac{dy}{y} \Rightarrow -\frac{1}{xy} = -\log y + C$$

$$\Rightarrow -\frac{1}{xy} + \log y = C$$

10. Vector Algebra

1. (d) $\vec{v} \times \vec{w} = 3\hat{i} - 7\hat{j} - \hat{k}$

Now, $[\vec{u} \cdot \vec{v} \cdot \vec{w}] = \vec{u} \cdot (3\hat{i} - 7\hat{j} - \hat{k})$
 $= |\vec{u}| |3\hat{i} - 7\hat{j} - \hat{k}| \cos \theta$

(where θ is the angle between \vec{u} and $\vec{v} \times \vec{w}$)
 $= \sqrt{59} \cos \theta$

Thus $[\vec{u} \cdot \vec{v} \cdot \vec{w}]$ is maximum if $\cos \theta = 1$ i.e.
 $\theta = 0$ or $\vec{u} = \frac{1}{\sqrt{59}}(3\hat{i} - 7\hat{j} - \hat{k})$

2. (d) The given expression

$$\begin{aligned} &= \left\{ (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{b} \times \vec{c}) \right\} \cdot (\vec{b} + \vec{c}) \\ &= \left\{ (\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{c}) \times (\vec{b} \times \vec{a}) \times (\vec{b} \times \vec{c}) \right\} \cdot (\vec{b} + \vec{c}) \\ &= \left[(\vec{a} \cdot (\vec{b} \times \vec{c})) \vec{c} - (\vec{c} \cdot (\vec{b} \times \vec{c})) (\vec{b} \times \vec{c}) + (\vec{b} \cdot (\vec{b} \times \vec{c})) \vec{a} - (\vec{a} \cdot (\vec{b} \times \vec{c})) \vec{b} \right] \cdot (\vec{b} + \vec{c}) \\ &= \left[(\vec{a} \cdot (\vec{b} \times \vec{c})) (\vec{c} - \vec{b}) \cdot (\vec{b} + \vec{c}) \right] \\ &= (\vec{a} \cdot (\vec{b} \times \vec{c})) \left[|\vec{c}|^2 - |\vec{b}|^2 \right] = 0 \quad [|\vec{b}| = |\vec{c}| = 1] \end{aligned}$$

3. (a) Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$

Now, $\vec{a} \times \vec{b} = \vec{c}$ (Given)

$$\begin{aligned} &\Rightarrow \vec{a} \times (\vec{a} \times \vec{b}) = \vec{a} \times \vec{c} \Rightarrow (\vec{a} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{a}) \vec{b} = \vec{a} \times \vec{c} \\ &\Rightarrow 3\vec{a} - 3\vec{b} = -2\hat{i} + \hat{j} + \hat{k} \Rightarrow 3(\hat{i} + \hat{j} + \hat{k}) - 3\vec{b} = -2\hat{i} + \hat{j} + \hat{k} \\ &\Rightarrow 3\hat{i} + 3\hat{j} + 3\hat{k} - 3\vec{b} = -2\hat{i} + \hat{j} + \hat{k} \\ &\Rightarrow \vec{b} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k}) \Rightarrow |\vec{b}| = \frac{\sqrt{25+4+4}}{3} \Rightarrow |\vec{b}| = \sqrt{\frac{11}{3}} \end{aligned}$$

4. (c) Given $(\vec{a} + 3\vec{b}) \cdot (7\vec{a} + 5\vec{b}) = 0$

$$\Rightarrow 7\vec{a}^2 - 15\vec{b}^2 + 16 \vec{a} \cdot \vec{b} = 0 \quad \dots(1)$$

Also, $(\vec{a} - 4\vec{b}) \cdot (7\vec{a} - 2\vec{b}) = 0$

$$\Rightarrow 7\vec{a}^2 + 8\vec{b}^2 - 30 \vec{a} \cdot \vec{b} = 0 \quad \dots(2)$$

Subtracting, $-23\vec{b}^2 + 46 \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \cdot \vec{b} = \frac{\vec{b}^2}{2}$

Putting this in (1),

$$7\vec{a}^2 - 7\vec{b}^2 = 0 \Rightarrow |\vec{a}| = |\vec{b}|$$

Thus $\vec{a} \cdot \vec{b} = ab \cos \theta$

$$\Rightarrow \frac{\vec{b}^2}{2} = \vec{b}^2 \cos \theta \Rightarrow \cos \theta = \frac{1}{2}$$

or $\theta = 60^\circ$.

5. (b) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\Rightarrow \vec{a} \times \hat{i} = -a_2\hat{k} + a_3\hat{j}$$

$$\Rightarrow |\vec{a} \times \hat{i}|^2 = a_2^2 + a_3^2$$

$$\begin{aligned} \therefore \text{L.H.S.} &= (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2) \\ &= 2|\vec{a}|^2 = 2a^2. \end{aligned}$$

6. (b) Since $[\vec{b} + (\vec{a} \times \vec{b})] \cdot \vec{a} = 0$ and $[\vec{b} + (\vec{a} \times \vec{b})] \cdot \vec{b} = 1$
So, $\vec{r} = \vec{b} + (\vec{a} \times \vec{b})$

7. (d) We have, $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$
 $\Rightarrow (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$

On comparing, $\vec{a} \cdot \vec{b} = \frac{\sqrt{3}}{2}$. Then $\theta = \frac{5\pi}{6}$

8. (a) The direction cosines of the two lines are given by
 $al + bm + cn = 0 \quad \dots(i)$
 $fmn + gnl + hlm = 0 \quad \dots(ii)$

From (i)

$$n = -\left(\frac{al + bm}{c}\right)$$

Putting this value of n in (ii), we have

$$\Rightarrow fm\left(\frac{al + bm}{c}\right) - gl\left(\frac{al + bm}{c}\right) + hlm = 0$$

$$\Rightarrow fm(al + bm) + gl(al + bm) - hclm = 0$$

$$\Rightarrow agl^2 + lm(ag + bg - ch) + fbm^2 = 0$$

Dividing throughout by m^2 , we have

$$ag \frac{l^2}{m^2} + \frac{1}{m}(af + bg - ch) + bg = 0 \quad \dots(iii)$$

If l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c's of the two lines, then

$\frac{l_1}{m_1}, \frac{l_2}{m_2}$ are the roots of the equation (iii)

$$\therefore \frac{l_1 l_2}{m_1 m_2} = \frac{bf}{ag} = \text{Product of the roots}$$

$$= \frac{f/a}{g/b}$$

$$\Rightarrow \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} = k \quad (\text{say})$$

the two lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$$\Rightarrow k\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right) = 0$$

$$\Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$$

9. (d) $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c})$

$$= (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) - (\vec{d} \times \vec{b}) + (\vec{d} \times \vec{c})$$

$$\begin{aligned}
&= (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{d}) + (\vec{d} \times \vec{c}) \\
&= (\vec{a} \times \vec{b}) - (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{d}) - (\vec{a} \times \vec{c}) = 0
\end{aligned}$$

So, $(\vec{a} - \vec{d})$ and $(\vec{b} - \vec{c})$ are parallel.

- 10. (a)** $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + 4\hat{k}$, $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$
 \vec{a} and \vec{c} are orthogonal $\Rightarrow \vec{a} \cdot \vec{c} = 0$ giving $\lambda - 1 + 2\mu = 0$
Also \vec{b} and \vec{c} are orthogonal $\Rightarrow 2\lambda + 4 + 4\mu = 0$
Solving the equations we get $\lambda = -3$, $\mu = 2$

11. (d) $|\vec{a} \times \vec{b}| \cdot |\vec{c}| = |(\vec{a} \cdot |\vec{b}| \sin \theta) \cdot |\vec{c}| \hat{k}|$
 $= |\vec{a}| |\vec{b}| |\vec{c}| \sin \theta |\hat{n}| |\hat{k}| \cos \phi$
 $= |\vec{a}| |\vec{b}| |\vec{c}| |\hat{n}| |\hat{k}| \sin \theta \cos \phi \quad \dots(i)$

From (i),

$$|\vec{a}| |\vec{b}| |\vec{c}| |\hat{n}| |\hat{k}| \sin \theta \cos \phi = |\vec{a}| |\vec{b}| |\vec{c}|$$

$$\Rightarrow \sin \theta \cos \phi = 1$$

Clearly,

$$\theta = \frac{\pi}{2} \text{ and } \phi = 0$$

12. (c) $\vec{AB} = 5\hat{i} - \hat{j} + 9\hat{k}$

$$\vec{CD} = 2\hat{i} + 3\hat{j} - 6\hat{k}$$

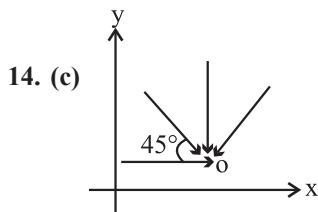
Projection of \vec{AB} on \vec{CD} is given by

$$\frac{\vec{AB} \cdot \vec{CD}}{|\vec{AB}|} = \frac{(5\hat{i} - \hat{j} + 9\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 6\hat{k})}{\sqrt{49}} = \frac{-47}{7}$$

13. (c) $[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix} \quad C_3 \rightarrow C_3 + C_1 = \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} = 1(1) = 1$$

which is independent of x and y.



$$R_x \text{ (Net horizontal component)} = k + \frac{k}{\sqrt{2}} - \frac{k}{\sqrt{2}}$$

$$R_y \text{ (Net vertical component)} = -\left(k + \frac{k}{\sqrt{2}} + \frac{k}{\sqrt{2}}\right)$$

Therefore,

$$\text{Resultant } (R) = k \hat{i} - (k + \sqrt{2}k) \hat{j}$$

$$|R| = \sqrt{k^2 + k^2(3 + 2\sqrt{2})}$$

$$= k\sqrt{4 + 2\sqrt{2}}$$

15. (b) Let use Coordinate Geometry for this one.

Let A be the origin (0, 0), B = (4m, 0), and C = (4p, 4q).

The midpoint of BC, CA, and AB is M_1 , M_2 , and M_3 respectively. So,

$$M_1 = (2m + 2p, 2q)$$

$$M_2 = (2p, 2q)$$

$$M_3 = (2m, 0)$$

Then, each median of a triangle divides the median in the ratio 1:3 reckoning from the vertex.

Let E, F, and G denotes the points on each median, using the ratio formula:

$$(x, y) = \left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right], \text{ with ratio m:n. Hence,}$$

$$E = \left[\frac{2m+2p+0}{4}, \frac{2q+0}{4} \right] = \left(\frac{m+p}{2}, \frac{q}{2} \right)$$

$$F = \left[\frac{2p+12n}{4}, \frac{2q+0}{4} \right] = \left(\frac{p+6n}{2}, \frac{q}{2} \right)$$

$$G = \left[\frac{2m+12p}{4}, \frac{0+12q}{4} \right] = \left(\frac{n+6p}{2}, 3q \right)$$

$$\text{Area } \Delta ABC = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

$$= \frac{1}{2} |0 + 16mq + 0 - 0 - 0 - 0| = 8mq$$

$$\text{Area } \Delta EFG = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|$$

$$= \frac{1}{2} \left| \frac{mq + pq}{4} + \frac{3pq + 18mg}{2} + \frac{mg + 6pq}{4} \right| - \frac{(pq + 6mg)}{4} - \frac{(mq + 6pq)}{4} - \frac{(3mq + 3pq)}{2} = \frac{25mq}{8}$$

$$\text{So, } \frac{\text{Ar. } \Delta EFG}{\text{Ar. } \Delta ABC} = \frac{25mg}{8} \times \frac{1}{8mq} = \frac{25}{64}.$$

16. (a) Volume $= (\vec{a} \times \vec{b}) \cdot \vec{c} = 7$

17. (b) Force $= (2\hat{i} - 5\hat{j} + 6\hat{k}) + (-\hat{i} + 2\hat{j} - \hat{k}) + (2\hat{i} + 7\hat{j}) = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$$\text{Displacement} = 2\hat{i} + 4\hat{j} - \hat{k}$$

$$\text{Work done} = \text{Force} \cdot \text{Displacement} =$$

$$(3\hat{i} + 4\hat{j} + 5\hat{k}) \cdot (2\hat{i} + 4\hat{j} - \hat{k}) = 17 \text{ units}$$

18. (d) Given,

$$|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$$

Also,

each one of them being perpendicular to the sum of the

other two

\vec{a} is perpendicular to $\vec{b} + \vec{c}$

$$\text{i.e., } \vec{a} \cdot (\vec{b} + \vec{c}) = 0$$

$$= \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{c} = 0 \quad \dots(1)$$

\vec{b} is perpendicular to $\vec{a} + \vec{c}$

$$\text{i.e., } \vec{b} \cdot (\vec{a} + \vec{c}) = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{c} = 0 \quad \dots(2)$$

\vec{c} is perpendicular to $\vec{a} + \vec{b}$

$$\text{i.e., } \vec{c} \cdot (\vec{a} + \vec{b}) = 0$$

$$\Rightarrow \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0 \quad \dots(3)$$

Now,

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} \\ &= \vec{a} \cdot \vec{a} + (\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}) + \vec{b} \cdot \vec{b} + (\vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{c}) + \vec{c} \cdot \vec{c} + (\vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b}) \\ &= \vec{a} \cdot \vec{a} + (0) + \vec{b} \cdot \vec{b} + (0) + \vec{c} \cdot \vec{c} + (0) \quad (\text{From (1), (2), (3)}) \\ &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \quad (\text{Using prop: } \vec{a} \cdot \vec{a} = |\vec{a}|^2) \\ &= 3^2 + 4^2 + 5^2 \\ &= 9 + 16 + 25 \\ &= 50 \end{aligned}$$

So,

$$|\vec{a} + \vec{b} + \vec{c}|^2 = 50$$

Taking square root both sides,

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{50}$$

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{25} \times \sqrt{2}$$

$$|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$$

$$\text{So, } |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$$

$$19. (b) \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\text{So, } |\vec{a} + \vec{b} + \vec{c}| = |\vec{0}| = 0$$

Now,

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} \\ &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} \end{aligned}$$

$$\text{Using prop: } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\begin{aligned} &= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} \\ &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ &\text{Prop: } \vec{a} \cdot \vec{a} = |\vec{a}|^2 \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \end{aligned}$$

$$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = 3^2 + 1^2 + 4^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$\Rightarrow 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) + 26 = 0$$

$$\Rightarrow (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -13$$

$$20. (a) (\vec{u} + \vec{v} - \vec{w}) \cdot [\vec{u} \times \vec{v} - \vec{u} \times \vec{w} + \vec{v} \times \vec{w}]$$

$$\because \vec{v} \times \vec{v} = 0$$

$$\begin{aligned} &\vec{u} \cdot (\vec{u} \times \vec{v}) - \vec{u} \cdot (\vec{u} \times \vec{w}) + \vec{u} \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot (\vec{u} \times \vec{v}) - \vec{v} \cdot (\vec{u} \times \vec{w}) \\ &+ \vec{v} \cdot (\vec{v} \times \vec{w}) - \vec{w} \cdot (\vec{u} \times \vec{v}) + \vec{w} \cdot (\vec{u} \times \vec{w}) - \vec{w} \cdot (\vec{v} \times \vec{w}) \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot (\vec{w} \times \vec{u}) - \vec{w} \cdot (\vec{u} \times \vec{v}) \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) + \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot (\vec{v} \times \vec{w}) \\ &= \vec{u} \cdot (\vec{v} \times \vec{w}) \quad (\because [\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = [\vec{c} \vec{a} \vec{b}]) \end{aligned}$$

11. Three Dimensional Geometry

- 1. (c)** Let the direction ratios of the plane be a, b and c .

Then, $a = -3, b = 2$ and $c = 1$ [given]

The plane contain the line $3x + 2y + c = 0$ (1)

Equation of plane is given as

$$a(x - 0) + b(y - 7) + c[z - (-7)] = 0$$

$$\Rightarrow a(x - 0) + b(y - 7) + c(z + 7) = 0$$

The point $(-1, 3, -2)$ satisfy the equation of plane.

$$\Rightarrow a(-1 - 0) + b(3 - 7) + c(-2 + 7) = 0$$

$$\Rightarrow a + 4b - 5c = 0$$

.... (2)

On solving (1) and (2) we get

$$a = 7, b = -8, c = -5$$

Thus, equation of the plane is

$$7(x - 0) - 8(y - 7) - 5(z + 7) = 0$$

$$\Rightarrow 7x - 8y - 5z + 21 = 0$$

2. (c) $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{2}$

$$\Rightarrow x - y + z - 6 = 0$$

Perpendicular distance from the point $(1, 2, 3)$ to the given line is

$$\frac{|la + mb + nc + d|}{\sqrt{l^2 + m^2 + n^2}}$$

$$\Rightarrow \frac{|1(1) + (-1)(2) + 1(3) + 6|}{\sqrt{(1)^2 + (-1)^2 + (1)^2}}$$

$$= \frac{1 - 2 + 3 + 6}{\sqrt{3}} = \frac{8}{\sqrt{3}}$$

- 3. (d)** The given equations are

$$l + 3m + 5n = 0 \quad \dots(i) \quad \text{and} \quad 5lm - 2mn + 6nl = 0 \quad \dots(ii)$$

From (i), $l = -3m - 5n$

Putting this value of l in (ii), we have

$$5(-3m - 5n)m - 2mn + 6n(-3m - 5n) = 0$$

$$\Rightarrow -15m^2 - 30n^2 - 45mn = 0 \Rightarrow m^2 + 2n^2 + 3mn = 0$$

$$\Rightarrow m^2 + 3mn + 2n^2 = 0 \Rightarrow m(m + 2n) + n(m + 2n) = 0$$

$$\Rightarrow (m + n)(m + 2n) = 0 \Rightarrow \text{either } m = -n \text{ or } m = -2n$$

For $m = -n$, $l = -2n$; For $m = -2n$, $l = n$

\therefore Direction ratios of two lines are

$(-2n, -n, n)$ and $(n, -2n, n)$ i.e., $(-2, -1, 1)$ and $(1, -2, 1)$

$$\therefore \text{The required angle is } \cos \theta = \frac{-2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1}{\sqrt{4+1+1} \sqrt{1+4+1}}$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

- 4. (b)**

- 5. (c)**

- 6. (c)** The direction cosines of the normal to the given plane are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow l = \frac{2}{\sqrt{2^2 + 3^2 + (-6)^2}}, m = \frac{3}{\sqrt{2^2 + 3^2 + (-6)^2}}, n = \frac{-6}{\sqrt{2^2 + 3^2 + (-6)^2}}$$

$$\Rightarrow l = \frac{2}{7}, m = \frac{3}{7}, n = \frac{-6}{7}$$

- 7. (d)** Angle between the lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos \theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\text{Given, } a_1 = a, \quad b_1 = b, \quad c_1 = c$$

$$\text{and } a_2 = b - c, \quad b_2 = c - a, \quad c_2 = a - b$$

$$\text{So, } \cos \theta = \left| \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2 + b^2 + c^2} \sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} \right|$$

$$= \left| \frac{ab - ac + bc - ab + ca - bc}{\sqrt{a^2 + b^2 + c^2} \sqrt{b^2 + c^2 - 2bc + c^2 + a^2 - 2ca + a^2 + b^2 - 2ab}} \right| \\ = 0$$

$$\therefore \cos \theta = 0$$

$$\text{So, } \theta = 90^\circ$$

Therefore, angle between the given pair of lines is 90° .

- 8. (c)** Lines are coplanar

$$\therefore \begin{vmatrix} 3-1 & 2-2 & 1-(-3) \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & 0 & 4 \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix}$$

$$2(4 - \lambda^4) + 4(\lambda^2 - 2) = 0$$

$$4 - \lambda^4 + 2\lambda^2 - 4 = 0 \Rightarrow \lambda^2(\lambda^2 - 2) = 0 \Rightarrow \lambda = 0, \sqrt{2}, -\sqrt{2}$$

- 9. (c)** Equation of a plane parallel to xy -plane is $z = k$

$$\therefore z = 3$$

- 10. (a)** Since direction cosine, $n = 0$ so the given line is parallel to xy -plane.

- 11. (a)** From given lines $\frac{x}{3} = \frac{y}{2} = \frac{z}{-6}$ and $\frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos \theta = \frac{6 - 24 + 18}{\sqrt{3^2 + 2^2 + (-6)^2} \sqrt{2^2 + (-12)^2 + (-3)^2}} = 2 \therefore \theta = 90^\circ$$

- 12. (d)**

- 13. (d)** The required equations of the bisector planes are

$$\Rightarrow \frac{3x - 6y + 2z + 5}{\sqrt{3^2 + (-6)^2 + 2^2}} = \pm \frac{4x - 12y + 3z - 3}{\sqrt{4^2 + (-12)^2 + 3^2}}$$

$$\Rightarrow \frac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \pm \frac{4x - 12y + 3z - 3}{\sqrt{16 + 144 + 9}}$$

$$\Rightarrow \frac{3x - 6y + 2z + 5}{\sqrt{49}} = \pm \frac{4x - 12y + 3z - 3}{\sqrt{169}}$$

$$\Rightarrow \frac{3x - 6y + 2z + 5}{7} = \pm \frac{4x - 12y + 3z - 3}{13}$$

$$\Rightarrow 39x - 78y + 26z + 65 = \pm 28x - 84y + 21z - 21$$

$$\Rightarrow 11x + 6y + 5z + 86 = 0 \quad \text{OR}$$

$$\Rightarrow 39x - 78y + 26z + 65 = -28x + 84y - 21z + 21$$

$$\Rightarrow 67x - 162y + 47z + 44 = 0$$

14. (a) The given lines are coplanar if

$$0 = \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 \\ k & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ k & k+2 & 1+k \end{vmatrix}$$

$$\text{or if } 2(1+k) - (k+2)(1-k) = 0 \text{ or if } k^2 + 3k = 0$$

$$\text{or if } k = 0, -3.$$

15. (b) Let $P(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the origin O on the given plane. The plane passes through $A(a, b, c)$. Therefore,

$$\overrightarrow{AP} \perp \overrightarrow{OP} \Rightarrow \overrightarrow{AP} \cdot \overrightarrow{OP} = 0$$

$$\Rightarrow [(\alpha - a)\hat{i} + (\beta - b)\hat{j} + (\gamma - c)\hat{k}] \cdot (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) = 0$$

$$\Rightarrow \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$$

Hence, the locus of (α, β, γ) is

$$x_2 + y_2 + z_2 - ax - by - cz = 0$$

Clearly, it is a sphere of radius $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$

16. (b) $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$

$$[\vec{a} \vec{b} \vec{c}] = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0 \Rightarrow 1(1-2x+4) - 1(-1-2x) + 1(x-2+x) = 0$$

$$\Rightarrow 5 - 2x + 1 + 2x + 2x - 2 = 0 \Rightarrow x = -2$$

17. (b) Let image of the point $P(1, 3, 4)$ in the given plane be the point Q. The equation of the line through

$$P \text{ and normal to the given plane is } \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$$

Since this line passes through Q, so let the coordinates of Q be $(2r+1, -r+3, r+4)$.

The coordinates of the mid-point of PQ are $\left(r+1, -\frac{r}{2}+3, \frac{r}{2}+4\right)$

This point lies on the given plane. Therefore, $r = -2$.

Hence (b) is the correct answer because the coordinates of Q are $(-3, 5, 2)$.

18. (b) If the given plane contains the given line, then normal to the plane must be perpendicular to the line and the condition for the same is $a\ell + bm + cn = 0$.

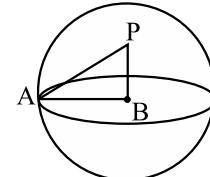
19. (b) The radius and centre of sphere

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0 \text{ is}$$

$$\sqrt{2^2 + 1^2 + 4 + 19} = 5 \text{ and centre } (-1, 1, 2)$$

$PB \perp$ from centre to the plane

$$\frac{|-1+2+4+7|}{\sqrt{1+2^2+2^2}} = 4$$



$$\text{Now } (AB)^2 = AP^2 - PB^2 = 25 - 16 = 9 \therefore AB = 3$$

20. (a) $\vec{r} = 2\hat{i} + 3\hat{j} - 6\hat{k}$

$$\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$$

12. Probability

$$\begin{aligned}
 1. \text{ (a)} \quad & P(B|A \cup B') = \frac{P(B \cap (A \cup B'))}{P(A \cup B')} \\
 &= \frac{P(A \cap B)}{P(A) + P(B') - P(A \cap B')} = \frac{P(A) - P(A \cap B')}{0.7 + 0.6 - 0.5} \\
 &= \frac{0.7 - 0.5}{0.8} = \frac{1}{4}
 \end{aligned}$$

2. (b) Let E be the event that a television chosen randomly is of standard quality. We have to find

$$\begin{aligned}
 P(II|E) &= \frac{P(E|II).P(II)}{P(E|I).P(I) + P(E|II).P(II)} \\
 &= \frac{\frac{9}{10} \times \frac{3}{10}}{\frac{4}{5} \times \frac{7}{10} + \frac{9}{10} \times \frac{3}{10}} = \frac{27}{83}
 \end{aligned}$$

3. (d) We know that variance = npq

$$\begin{aligned}
 P(\text{probability of drawing a green ball}) &= \frac{15}{25} = \frac{3}{5} \\
 \text{Here, } n &= 10, p = \frac{3}{5}, q = \frac{2}{5} \text{ then, variance} = 10 \cdot \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{5}
 \end{aligned}$$

4. (d) Let $P(F) = p \Rightarrow P(S) = 2p$

$$\begin{aligned}
 \text{Now, } p + 2p &= 1 \Rightarrow p = \frac{1}{3} \therefore P(x \geq 5) = P(x = 5) + P(x = 6) \\
 &= {}^6C_5 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^5 + {}^6C_6 \left(\frac{2}{3}\right)^6 = \frac{256}{729}
 \end{aligned}$$

5. (d) Total number of subsets of X = 2^{10}

$$\text{Required probability} = \frac{{}^{10}C_0^2 + {}^{10}C_1^2 + {}^{10}C_2^2 + \dots + {}^{10}C_{10}^2}{(2^{10})^2} = \frac{{}^{20}C_{10}}{2^{20}}$$

6. (d) $P(\text{correct answer}) = \frac{1}{3}$

$$\begin{aligned}
 \text{The required probability} &= {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) + {}^5C_5 \left(\frac{1}{3}\right)^5 \\
 &= \frac{5 \times 2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}
 \end{aligned}$$

7. (d) Given $np = 4$ and $npq = 2$

$$q = \frac{npq}{np} = \frac{2}{4} = \frac{1}{2} \text{ so } p = 1 - \frac{1}{2} = \frac{1}{2}$$

Now $npq = 2 \therefore n = 8$

\therefore BD is given by $P(X = r) = {}^8C_r p^r q^{8-r}$

$$\therefore P(X = r = 2) = {}^8C_2 \left(\frac{1}{2}\right)^8 = \frac{28}{256}$$

8. (b) $P(A) = \text{number is } 4$

$P(B) = \text{sum is } 6$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

9. (c) Total case

$$= (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (1,5) (2,5) (3,5) (4,5) (6,5) = 11$$

Favourable case = (5,5) (5,6) (6,5) = 3

$$\text{Required probability} = \frac{3}{11}$$

10. (c) Given $P(A) = 0.4$ and $P\left(\frac{B}{A}\right) = 0.5$

$$\therefore P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = 0.5 \times 0.4 = 0.2$$

$$\therefore P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - 0.2 = 0.8$$

11. (a) $P(A) = \frac{2}{4} = \frac{1}{2}$

$$P(B) = \frac{3}{4}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/2}{1/2} = 1$$

12. (b) $p = \frac{2}{6} = \left(\frac{1}{3}\right), q = \frac{4}{6} = \left(\frac{2}{3}\right)$

2 success + 3 success

$$\begin{aligned}
 {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 + {}^3C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 \\
 = \frac{6}{27} + \frac{1}{27} = \frac{7}{27}
 \end{aligned}$$

13. (d) ${}^{2n+1}C_1 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^{2n} + {}^{2n+1}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{2n-2} +$

$$\dots \dots + {}^{2n+1}C_{2n+1} \left(\frac{1}{2}\right)^{2n+1}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{2n+1} [C_1 + C_3 + C_5 + \dots + C_{2n+1}] = C_0 + C_1 + C_2 + \dots + C_{2n+1} = 2^{2n+1} \quad \dots(1)$$

$$C_0 - C_1 + C_2 + \dots + C_{2n+1} = 0 \quad \dots(2)$$

$$(1) - (2)$$

$$\Rightarrow C_1 + C_3 + \dots + C_{2n+1} = 2^{2n}$$

$$\left(\frac{1}{2}\right)^{2n+1} (2^{2n}) = \frac{1}{2}$$

14. (c)

15. (b) 

$$P(E_1) = \frac{1}{3} = P(E_2) = P(E_3)$$

$$P(A/E_1) = \frac{2}{5}, \quad P(A/E_3) = \frac{4}{5}$$

$$P(A/E_2) = \frac{3}{5}$$

$$P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{4}{5}} = \frac{2}{9}$$

16. (a) $P(E_1) = \frac{25}{100}, \quad P(A/E_1) = \frac{5}{100}$

$$P(E_2) = \frac{35}{100}, \quad P(A/E_2) = \frac{4}{100}$$

$$P(E_3) = \frac{40}{100}, \quad P(A/E_3) = \frac{2}{100}$$

$$P\left(\frac{E_1}{A}\right) = \frac{25 \times 5}{125 + 140 + 80} = \frac{125}{345}$$

$$P\left(\frac{E_2}{A}\right) = \frac{140}{345} = \frac{28}{69}$$

$$P\left(\frac{E_3}{A}\right) = \frac{80}{345} = \frac{16}{69}$$

17. (a)

18. (b)

19. (c)

20. (c) Probability of drawing a white ball is $\frac{1}{2}$

Probability of not drawing a white ball is $\frac{1}{2}$

We want to draw a white ball 4th time in 7th draw

So a white ball drawn in 7th draw and 3 white balls are drawn in first 6 draws

So required probability

$$= {}^6C_3 p^3 q^3 \cdot p$$

$$= {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \frac{5}{32}$$